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A new method for ordering triangular fuzzy numbers

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Abstract

Ranking fuzzy numbers plays a very important role in linguistic decision making and other fuzzy application systems. In spite of many ranking methods, no one can rank fuzzy numbers with human intuition consistently in all cases. Shortcoming are found in some of the convenient methods for ranking triangular fuzzy numbers such as the coefficient of variation (CV index), distance between fuzzy sets, centroid point and original point, and also weighted mean value. In this paper, we introduce a new method for ranking triangular fuzzy number to overcome the shortcomings of the previous techniques. Finally, we compare our method with some convenient methods for ranking fuzzy numbers to illustrate the advantage our method.

Keywords: Linear order, ranking fuzzy numbers, triangular fuzzy number.

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1. Introduction

Ranking fuzzy number plays a very important role in the most decision making problems. Most than 20 fuzzy ranking indices have been proposed since 1976 [9,10]. Various techniques are applied in the literature to compare fuzzy numbers [1-8,11,12]. Some of these methods compared and reviewed by Bortolan and Degani [3] (see also [15]). Lee et al. [12] ranked fuzzy numbers based on two different criteria, namely, the fuzzy mean and the fuzzy spread of fuzzy numbers. They pointed that human intuition world favor a fuzzy number with the following characteristics: higher mean value and at the same time lower spread. However, when higher mean value and at the same time higher spread or lower mean value and at the same time lower spread exists, it is not easy to compare its ordering clearly. Therefore, Cheng [6] proposed the coefficient of variance (CV index) to improve Lee and Li's ranking method [12]. Chu [7] pointed out the shortcomings of Cheng's method and suggested to rank fuzzy numbers with the area between the centroid point and the point of origin. In this paper, we introduce a new method for ranking fuzzy numbers to overcome the shortcomings of the previous techniques. Finally, we compare our method with pioneering methods for ranking fuzzy numbers to illustrate the advantage our method.

This paper is organized in 5 Sections. In Section 2, we give some basic definitions of fuzzy sets theory. In Section 3, we introduce a new ranking technique for triangular fuzzy numbers. In Section 4, we use some examples to show the advantage of the proposed method. We conclude in Section 5.

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1. Fuzzy Numbers and Fuzzy Arithmetic

1.1. Fuzzy numbers

Here, we first give some necessary definitions and notations of fuzzy set theory.

Definition 2.1: Fuzzy set s and membership functions. If X is a collection of objects denoted generically by x , then a **fuzzy set** A in X is defined to be a set of ordered pairs $A = \{(x, \mu_A(x)) | x \in X\}$, where $\mu_A(x)$ is called the **membership function** for the fuzzy set. The membership function maps each element of X to a membership value between 0 and 1.

Remark 2.1: We assume that X is the real line $\mathbb{R}$.

Definition 2.2: Support. The **support** of a fuzzy set A is the set of points x in X with $\mu_A(x) > 0$.

Definition 2.3: Core. The **core** of a fuzzy set is the set of points x in X with $\mu_A(x) = 1$.

Definition 2.4: Normality. A fuzzy set A is called **normal** if its core is nonempty. In other words, there is at least one point $x \in X$ with $\mu_A(x) = 1$.

Definition 2.5: α −cut and **strong** α −cut. The α −cut or α −level set of a fuzzy set A is a crisp set defined by $A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$. The **strong** α −cut is defined to be $\overline{A}_\alpha = \{x \in X | \mu_A(x) > \alpha\}$.

Definition 2: Convexity. A fuzzy set A on X is **convex** if for any $x, y \in X$ and any $\lambda \in [0,1]$, we have $\mu_A(\lambda x + (1-\lambda)y) \geq min\{\mu_A(x), \mu_A(y)\}$.

Remark 2.2: A fuzzy set is convex if and only if all its α −cuts are convex.

Definition 2.7: Fuzzy number. A **fuzzy number** A is a fuzzy set on the real line that satisfies the conditions of normality and convexity.

Definition 2.8: A fuzzy number $\tilde{a}$ an $\mathbb{R}$ is said to be a triangular fuzzy number, if there exist real numbers a^m and $a^\alpha, a^\beta \geq 0$ (at least one is not zero) such that

$$\tilde{a}(x) = \begin{cases} \dfrac{x}{a^\alpha} + \dfrac{a^\alpha - a^\beta}{a^\alpha}, & x \in [a^m - a^\alpha, a^m] \\ \dfrac{-x}{a^\beta} + \dfrac{a^m + a^\beta}{a^\beta}, & x \in [a^m, a^m + a^\beta] \\ 0, & o.w. \end{cases} \qquad (2-1)$$

We denote a triangular fuzzy number $\tilde{a}$ by $\tilde{a} = (a^m, a^\alpha, a^\beta)$, where the support of $\tilde{a}$ is $(a^m - a^\alpha, a^m + a^\beta)$ and we denote the set of all triangular fuzzy numbers with $F(\mathbb{R})$.

Definition 2: A triangular fuzzy number $\tilde{a} = (a^m, a^\alpha, a^\beta)$ is called symmetric, if $a^\alpha = a^\beta$.

1.2. Arithmetic on triangular fuzzy numbers

Let $\tilde{a} = (a^m, a^\alpha, a^\beta)$ and $\tilde{b} = (b^m, b^\alpha, b^\beta)$ be two triangular fuzzy numbers and $\in \mathbb{R}$. Define:

$$x \geq 0, \quad x\tilde{a} = (xa^m, xa^\alpha, xa^\beta),$$
$$x < 0, \quad x\tilde{a} = (xa^m, -xa^\beta, -xa^\alpha),$$

$\tilde{a}+\tilde{b}=\left(a^m+b^m, a^\alpha+b^\alpha, a^\beta+b^\beta\right).$

2. Construction of a new method for ranking of fuzzy number

Here, we construct a new ranking system for triangular fuzzy numbers which is very realistic and efficient and then introduce a new algorithm for ranking triangular fuzzy numbers.

For any triangular fuzzy number $\tilde{a}=\left(a^m, a^\alpha, a^\beta\right)$, define

$$\underline{a}=a^m-\frac{1}{2}h_{\underline{a}},\quad \overline{a}=a^m+\frac{1}{2}h_{\overline{a}},\qquad (3-1)$$

Where $h_{\underline{a}}=\frac{a^\alpha}{a^\alpha+a^\beta}$ and $h_{\overline{a}}=\frac{a^\beta}{a^\alpha+a^\beta}$.

Now assume that $\tilde{a}=\left(a^m, a^\alpha, a^\beta\right)$, $\tilde{b}=\left(b^m, b^\alpha, b^\beta\right)$ be two triangular fuzzy numbers, Let

$$\begin{aligned}\overline{R}(\tilde{a},\tilde{b})&=\overline{a}-\overline{b},\\ \underline{R}(\tilde{a},\tilde{b})&=\underline{a}-\underline{b},\end{aligned}\qquad (3-2)$$

Where $\underline{a},\overline{a},\underline{b},\overline{b}$ are defined in (3-1).

Lemma 3.1 Assume $\tilde{a}=\left(a^m, a^\alpha, a^\beta\right)$, $\tilde{b}=\left(b^m, b^\alpha, b^\beta\right)$ be two triangular fuzzy numbers. Then, we have

$$\begin{aligned}\overline{R}(\tilde{a},\tilde{b})&=-\overline{R}(\tilde{b},\tilde{a})=\overline{R}(-\tilde{b},-\tilde{a})\\ \underline{R}(\tilde{a},\tilde{b})&=-\underline{R}(\tilde{b},\tilde{a})=\underline{R}(-\tilde{b},-\tilde{a})\end{aligned}\qquad (3-3)$$

Proof. It is straightforward from (3-2) . ■

Definition 3.1 Assume $\tilde{a}=\left(a^m, a^\alpha, a^\beta\right)$, $\tilde{b}=\left(b^m, b^\alpha, b^\beta\right)$ be two fuzzy numbers and $\underline{R}(\tilde{b},\tilde{a})\geq 0$. Define the relations $\prec$ and $\approx$ on $F(\mathbb{R})$ as given below:

i) $\tilde{a}\approx\tilde{b}$ if and only if $\underline{R}(\tilde{b},\tilde{a})=\overline{R}(\tilde{a},\tilde{b})$.

ii) $\tilde{a}\prec\tilde{b}$ if and only if $\underline{R}(\tilde{b},\tilde{a})>\overline{R}(\tilde{a},\tilde{b})$.

Remark 3.1 We denote $\tilde{a}\preccurlyeq\tilde{b}$ if and only if $\tilde{a}\approx\tilde{b}$ or $\tilde{a}\prec\tilde{b}$. Than $\tilde{a}\preccurlyeq\tilde{b}$ if and only if $\underline{R}(\tilde{b},\tilde{a})\geq\overline{R}(\tilde{a},\tilde{b})$. Also $\tilde{a}\prec\tilde{b}$ if and only if $\tilde{b}\succ\tilde{a}$.

We let $\tilde{0} = (0,0,0)$ as a zero triangular fuzzy numbers. Thus any $\tilde{a}$ such that $\tilde{a} \approx \tilde{0}$, is a zero too.

Lemma 3.2 Assume $\tilde{a} \prec \tilde{b}$, then $-\tilde{a} \succ -\tilde{b}$.

Proof. Since $\tilde{a} \prec \tilde{b}$, we have:

$$\underline{R}(\tilde{b}, \tilde{a}) > \overline{R}(\tilde{a}, \tilde{b})$$

So by use of Lemma 3.1, we have

$$\underline{R}(-\tilde{a}, -\tilde{b}) > \overline{R}(-\tilde{b}, -\tilde{a})$$

Now from Definition 3.1, we obtain $-\tilde{b} \prec -\tilde{a}$. ■

Lemma 3 Assume $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$. Then

i) $\tilde{a} \approx \tilde{a}$, for every $\tilde{a}$ (reflexivity),
ii) If $\tilde{a} \approx \tilde{b}$, then $\tilde{b} \approx \tilde{a}$ (symmetry),
iii) If $\tilde{a} \approx \tilde{b}$ and $\tilde{b} \approx \tilde{c}$, then $\tilde{a} \approx \tilde{c}$ (transitivity).

Proof. First part is obvious, because

$$\tilde{a} \approx \tilde{a} \Leftrightarrow \underline{R}(\tilde{a}, \tilde{a}) = \overline{R}(\tilde{a}, \tilde{a}) \Leftrightarrow \underline{a} - \underline{a} = \overline{a} - \overline{a}$$

Now for symmetry property, assume that $\tilde{a} \approx \tilde{b}$, then

$$\tilde{a} \approx \tilde{b} \Leftrightarrow \underline{R}(\tilde{b}, \tilde{a}) = \overline{R}(\tilde{a}, \tilde{b}) \Leftrightarrow \underline{b} - \underline{a} = \overline{a} - \overline{b}$$

Since we can rewrite $\underline{b} - \underline{a} = \overline{a} - \overline{b}$ as $\underline{a} - \underline{b} = \overline{b} - \overline{a}$, then

$$\tilde{a} \approx \tilde{b} \Leftrightarrow \underline{a} - \underline{b} = \overline{b} - \overline{a} \Leftrightarrow \underline{R}(\tilde{a}, \tilde{b}) = \overline{R}(\tilde{b}, \tilde{a}) \Leftrightarrow \tilde{b} \approx \tilde{a}$$

For transitivity property, assume that $\tilde{a} \approx \tilde{b}$ and $\tilde{b} \approx \tilde{c}$. Hence, from $\tilde{a} \approx \tilde{b}$ we have $\underline{R}(\tilde{a}, \tilde{b}) = \overline{R}(\tilde{b}, \tilde{a})$, or

$$\underline{b} - \underline{a} = \overline{a} - \overline{b} \qquad (3-4)$$

Also from $\tilde{b} \approx \tilde{c}$ we have $\underline{R}(\tilde{b}, \tilde{c}) = \overline{R}(\tilde{c}, \tilde{b})$, or

$$\underline{b} - \underline{c} = \overline{c} - \overline{b} \qquad (3-5)$$

Now from (3-4) and (3-5), we obtain

$$\underline{a} - \underline{c} = \overline{c} - \overline{a} \qquad (3-6)$$

Thus, $\underline{R}(\tilde{a}, \tilde{c}) = \overline{R}(\tilde{c}, \tilde{a})$ or equivalently we have $\tilde{a} \approx \tilde{c}$.

Remark 3 In fact, the above Lemma shows that the relation $\approx$ is an equivalence relation on $F(\mathbb{R})$.

More over, if $\tilde{a}$ is an element of $F(\mathbb{R})$, the fuzzy subset of $F(\mathbb{R})$ defined by $[a] = \{\tilde{b} \in F(\mathbb{R}) | \tilde{a} \approx \tilde{b}\}$ is called the equivalence fuzzy set $\tilde{a}$. The equivalent fuzzy set of $\tilde{a}$ is thus the set of all elements which are equivalent to $\tilde{a}$.

We now discuss the topic of order relations and denote this subject which is necessary for future works. The reader will find it helpful to keep in mind that a partial order relation is valid (as we prove it below) by Definition 3.1 on $F(\mathbb{R})$.

Lemma 3.4 Assume $\tilde{a}, \tilde{b} \in F(\mathbb{R})$. The relation $\preccurlyeq$ is a partial order on $F(\mathbb{R})$.

Proof. In fact, we need to prove the below triple properties.

i) $\tilde{a} \preccurlyeq \tilde{a}$, for every $\tilde{a}$ (reflexivity),
ii) If $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{b} \preccurlyeq \tilde{a}$, then $\tilde{a} \approx \tilde{b}$ (symmetry),
iii) If $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{b} \preccurlyeq \tilde{c}$, then $\tilde{a} \preccurlyeq \tilde{c}$ (transitivity).

The reflexivity property is valid, because

$$\tilde{a} \preccurlyeq \tilde{a} \Leftrightarrow \underline{R}(\tilde{a}, \tilde{a}) \geq \overline{R}(\tilde{a}, \tilde{a}) \Leftrightarrow \underline{a} - \underline{a} \geq \overline{a} - \overline{a}$$

For symmetry property, assume that $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{b} \preccurlyeq \tilde{a}$, then

$$\begin{cases} \tilde{a} \preccurlyeq \tilde{b} \Leftrightarrow \underline{R}(\tilde{b}, \tilde{a}) \geq \overline{R}(\tilde{a}, \tilde{b}) \Leftrightarrow \underline{b} - \underline{a} \geq \overline{a} - \overline{b} \\ \tilde{b} \preccurlyeq \tilde{a} \Leftrightarrow \underline{R}(\tilde{b}, \tilde{a}) \leq \overline{R}(\tilde{a}, \tilde{b}) \Leftrightarrow \underline{b} - \underline{a} \leq \overline{a} - \overline{b} \end{cases}$$

Or,

$$\begin{cases} \tilde{a} \preccurlyeq \tilde{b} \Leftrightarrow \underline{b} - \underline{a} \geq \overline{a} - \overline{b} \\ \tilde{b} \preccurlyeq \tilde{a} \Leftrightarrow \underline{b} - \underline{a} \leq \overline{a} - \overline{b} \end{cases}$$

Now since a natural partial order exists on $\mathbb{R}$, therefore it follows that $\underline{b} - \underline{a} = \overline{a} - \overline{b}$, or $\underline{R}(\tilde{a}, \tilde{b}) = \overline{R}(\tilde{b}, \tilde{a})$. Hence we obtain $\tilde{a} = \tilde{b}$.

Finally, for transitivity property, assume that $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{b} \preccurlyeq \tilde{c}$. So, since $\tilde{a} \preccurlyeq \tilde{b}$ we have:

$$\underline{R}(\tilde{b},\tilde{a}) \geq \overline{R}(\tilde{a},\tilde{b}),$$

Or equivalently,

$$\underline{b} - \underline{a} \geq \overline{a} - \overline{b} \qquad (3-7)$$

Also from $\tilde{b} \preccurlyeq \tilde{c}$, we have:

$$\underline{R}(\tilde{c},\tilde{b}) \geq \overline{R}(\tilde{b},\tilde{c}),$$

Or equivalently,

$$\underline{c} - \underline{b} \geq \overline{b} - \overline{c} \qquad (3-8)$$

From (3-7) and (3-8), we obtain

$$\underline{c} - \underline{a} \geq \overline{a} - \overline{c} \qquad (3-9)$$

Or

$$\underline{R}(\tilde{c},\tilde{a}) \geq \overline{R}(\tilde{a},\tilde{c})$$

It follows that $\tilde{a} \preccurlyeq \tilde{c}$. ■

Remark 3.3 We emphasis that the relation $\preccurlyeq$ is a linear order on $F(\mathbb{R})$ too, because any two elements in $F(\mathbb{R})$ are comparable by this relation.

Lemma 3 If $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{c} \preccurlyeq \tilde{d}$, then $\tilde{a} + \tilde{c} \preccurlyeq \tilde{b} + \tilde{d}$.

Proof. Since $\tilde{a} \preccurlyeq \tilde{b}$, we have:

$$\underline{R}(\tilde{b},\tilde{a}) \geq \overline{R}(\tilde{a},\tilde{b}),$$

Or equivalently,

$$\underline{b} - \underline{a} \geq \overline{a} - \overline{b} \qquad (3-10)$$

Also from $\tilde{c} \preccurlyeq \tilde{d}$ we have:

$$\underline{R}(\tilde{d},\tilde{c}) \geq \overline{R}(\tilde{c},\tilde{d}),$$

Or equivalently,

$$\underline{d} - \underline{c} \geq \overline{c} - \overline{d} \qquad (3-11)$$

From (3-10) and (3-11), we obtain

$$(\underline{b} + \underline{d}) - (\underline{a} + \underline{c}) \geq (\overline{a} + \overline{c}) - (\overline{b} + \overline{d}) \qquad (3-12)$$

Or

$$\underline{R}(\tilde{b} + \tilde{d}, \tilde{a} + \tilde{c}) \geq \overline{R}(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{d})$$

It follows that $\tilde{a} + \tilde{c} \preccurlyeq \tilde{b} + \tilde{d}$. ■

Now we can introduce our method for ordering all triangular fuzzy numbers by use of above discussion.

Algorithm 3.1 For two triangular fuzzy numbers $\tilde{a}$ and $\tilde{b}$, assume that $\underline{b} \geq \underline{a}$. Compute:

$\overline{R}(\tilde{a}, \tilde{b}) = \overline{a} - \overline{b}$ and $\underline{R}(\tilde{b}, \tilde{a}) = \underline{b} - \underline{a}$ (with $\underline{b} \geq \underline{a}$, it is obvious that $\underline{R}(\tilde{b}, \tilde{a}) \geq 0$).

Let $= \underline{R}(\tilde{b}, \tilde{a}) - \overline{R}(\tilde{a}, \tilde{b})$. Then

If $= 0$, then $\tilde{a} \approx \tilde{b}$,

If > 0 , then $\tilde{a} \prec \tilde{b}$, else $\tilde{b} \prec \tilde{a}$.

2. Numerical Examples

Here we compare the proposed method in the last section with some usual methods in the literature to illustrate the advantage our method.

Example 4.1 Consider the following triangular numbers as follows (taken from [1,2]):

$$A = (6,1,1), B = (6,0.1,1), C = (6,0,1).$$

The result of our method for ranking of the above triangular fuzzy numbers is: $A \prec B \prec C$. The results of ranking of the above triangular fuzzy numbers are given in Table 4.1. In Table 4.1, the results are as follows:

Table 4.1: Comparative results of Example 4.1

Fuzzy number	Asady	Choobineh and Lai	Chu and Tsao	Cheng distance	Chen
A	6	0.5	3	6.021	0.5
B	6.225	0.6125	3.126	6.349	0.5833
C	6.25	0.875	3.085	6.3519	0.5714
Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$	$A \prec C \prec B$

The result of Chu-Tsao method and Chen index is $A \prec C \prec B$, which is unreasonable. While the results of our method is similar to Cheng distance, Choobineh-Lai and Asady methods, i.e., $A \prec B \prec C$. In Fig. 4.1, we easily can verify it.

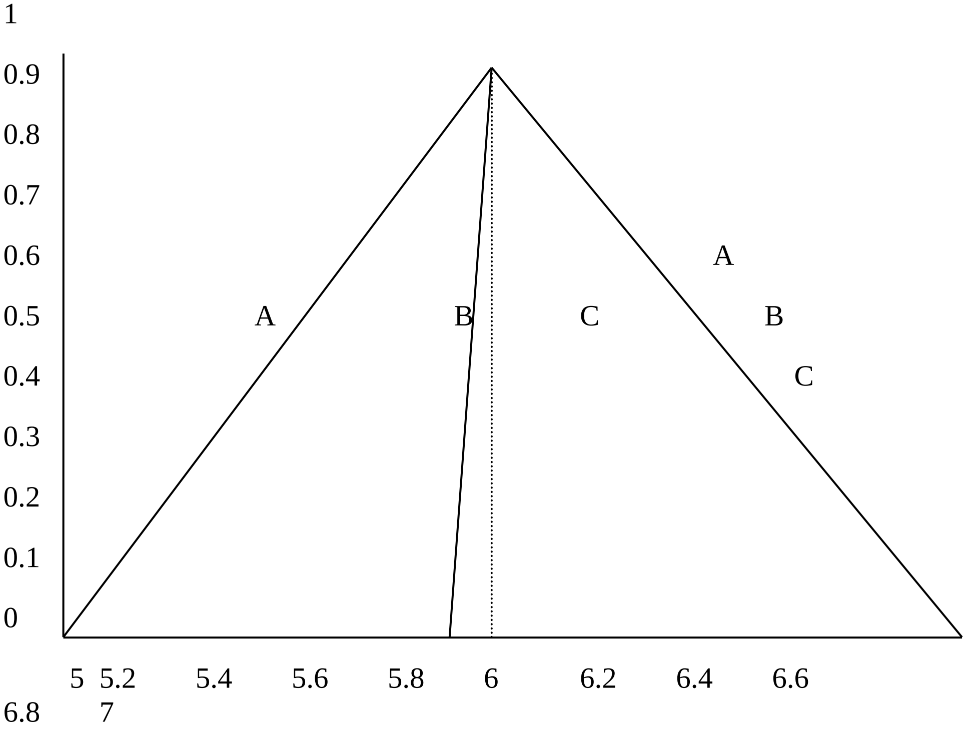


Fig. 4.1. The results from example 4.1

Example 4.2 Consider the following triangular fuzzy numbers as follows (taken from [1,2]):

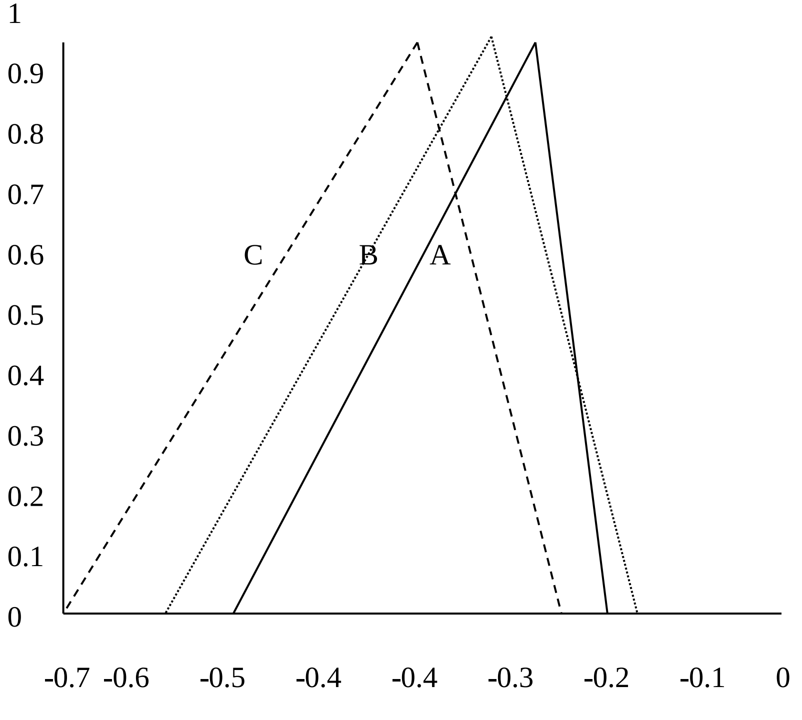
$A = (-0.3, 0.2, 0.1)$, $B = (-0.32, 0.26, 0.15)$, $C = (-0.4, 0.3, 0.15)$

The result of our method for ranking of the above triangular fuzzy numbers is: $A \succ B \succ C$. The results of ranking of the above triangular fuzzy numbers are given in Table 4.2. The results in Table 4.2. are as follows:

Table 4.2: Comparative results of Example 4.2

Fuzzy number	Asady	Choobineh and Lai	Chu and Tsao	Cheng distance	Chen
A	-0.325	0.536	-0.162	0.59	0.6708
B	-0.3475	0.504	-0.174	0.604	0.6302
C	-0.4375	0.375	-0.219	0.662	0.5116
Results	$C \prec B \prec A$	$C \prec B \prec A$	$C \prec B \prec A$	$A \prec B \prec C$	$C \prec B \prec A$

The result of Asady method, Chen method, Chu-Tsao and Choobineh-Lai methods is $A \succ B \succ C$ and their results are similar to our result. But the results of Cheng distance method is $A \prec B \prec C$. See in Fig. 4.2.



5. Conclusion

In this study, we constructed a new method for ordering triangular fuzzy numbers to overcome some shortcomings of pioneering approaches such as Chen method, Cheng distance and Chu-Tsao methods.

Finally, we gave some comparative examples to illustrate the advantage of our method. Also the proposed method will be useful for solving fuzzy linear programming problems by using ranking functions (see in [13,14]).

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