Numerical Optimal Control of The Wave Equation

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**Abstract**  
In this paper, we present a spectral method for approximating the boundary optimal control problems of a well-known wave equation by the linear optimal control problems. The method is based upon constructing the $M$th degree interpolation polynomials, using Chebyshevs nodes, to approximate the wave equation. Necessary conditions for optimal control functions are obtained by using the Pontryagin's maximum principle. Moreover, the control parameterization enhancing technique (CPET) is used to obtain the piecewise constant sub-optimal control functions. Finally, the efficiency of the proposed method is confirmed by a numerical example.

**Keywords:** Chebyshev, control parameterization, optimal control, spectral method; wave equation

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INRODUCTION
Consider a one-dimensional wave equation
\[ u_x(z,t) = \alpha^2 u_{zz}(z,t), \quad (z,t) \in (0, \ell) \times (0, \tau), \]
(1)
with the initial conditions,
\[ u(z,0) = u_1(z), \quad u_z(z,0) = u_2(z), \quad z \in (0, \ell), \]
(2)
and the boundary conditions,
\[ u(0,t) = \nu_1(t), \quad u(\ell,t) = \nu_2(t), \quad t \in (0, \tau), \]
(3)
and the end conditions,
\[ u(z,\tau) = \sigma_1(z), \quad u_z(z,\tau) = \sigma_2(z), \quad z \in (0, \ell), \]
(4)
where \( \nu_1 \) and \( \nu_2 \) are measurable control functions which are assumed to be constrained as
\[ \kappa_j \leq \nu_j(t) \leq \sigma_j, \quad j = 1, 2, \quad t \in (0, \tau) \]
(5)
Consider the problem of minimizing the functional
\[ J(\tau, \nu_1, \nu_2) = \int_0^\tau | \nu_1(t)|^p + | \nu_2(t)|^p \ dt, \]
(6)
subject to the constraints (1)-(5).
Optimal control problems of linear distributed parameter systems have been studied by many authors (Kamyad et al., 1991; Sakawa et al., 1999). A full discretization method based on the appropriate finite differences is used to solve a special case of this problem by Gerdts et al. (2008), where the functions \( s_1 \) and \( s_2 \) in the end conditions (4) are taken to be zero, the final time \( \tau \) is fixed and there are no constraints (5) on the control functions. Therefore, the problem considered in this paper is more general than the problem considered by Gerdts et al. (2008). Zarei and Bahrmand (2014) obtained an explicit solution for equations (1)-(3) and proposed a numerical method to solve a multi-objective optimal control of the wave equation. Optimal boundary control of the wave equation is studied by Farahi et al. (1996), using a measure theoretical approach. Moreover, the optimal control problems for the wave equation are studied by Manita (2008).

The spectral methods as an effective tool, have been used to solve the optimal control problems for lumped (Elnagar and Razzaghi, 1997) and, in recent years, distributed parameter systems (Chen et al., 2011a; 2011b; Zarei, 2015). In this paper, we use a spectral method to minimize the functional (6) subject to the constraints (1)-(5). The method is outlined in the next section. In section III, the distributed parameter system (1)-(4) is approximated by a lumped parameter system and the necessary conditions for optimal controls are derived, when \( p=1,2 \). The control parameterization enhancing technique (CPET) to obtain the sub-optimal control functions in piece-wise constant form is outlined in section IV. Section V includes the numerical results. The last section is devoted to the conclusion.

THE PROPOSED METHOD
Let \( T_{M}(x), \ x \in [-1,1] \) denotes the Chebyshev polynomial of degree \( M \), then the collocation points
\[ x_j = \cos \left( \frac{\pi j}{M} \right), \quad j = 0, 1, 2, \ldots, M, \]
(7)
are the zeros of \((1-x^2)T_{M}(x), \ x \in [-1,1]\) The \( M \)th degree interpolation polynomials to \( u(x,t) \) is given by
\[ u^M(x,t) = \sum_{j=0}^{M} a_j(t) \varphi_j(x), \]
(8)
where \( \varphi_j \) s are the Lagrange polynomials that
\[ \varphi_j(x_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases} \]
(9)
Le\( D = \left( d_{\ell k} \right) \) denotes the Chebyshev derivative matrix defined by
\[ d_{\ell k} = \begin{cases} \frac{c_\ell (-1)^{\ell-k}}{c_\ell (t_k - t_\ell)} & \text{if } k \neq \ell \\ \frac{2M^2 + 1}{6} & \text{if } k = \ell = 0 \\ -\frac{2M^2 + 1}{6} & \text{if } k = \ell = M \\ -\frac{t_\ell}{2(1-t_\ell^2)} & \text{if } 1 \leq k = \ell \leq M -1, \end{cases} \]
(10)
where, \( c_k = c_M = 2, c_i = 1, 1 \leq k \leq M - 1 \).

Then \( D^2 = \left( d^{(2)}_{kj} \right) \) is the second order Chebyshev derivative matrix which is used to compute the values of the function \( u_M^{(k)}(x, t) \) at the Chebyshev nodes \( x_k, k = 0, 1, 2, ..., M \) as

\[
u_M^{(k)}(x_k, t) = \sum_{j=0}^{M} d_{kj}^{(2)} a_j(t)
\]

In the next section we approximate the equations (1)-(4) by a linear control system, applying the proposed method.

**OPTIMAL CONTROL FORMULATION**

In order to use the Chebyshev nodes we introduce the transformation \( z = \frac{x}{2}(1 + x) \). In this way the equations (1)-(4) convert to

\[
u_s(x, t) = \beta^2 u_{ss}(x, t), (x, t) \in (-1,1) \times (0, \tau),
\]

with the initial conditions,
\[
u(x, 0) = u(x), \quad u_s(x, 0) = u_s(x), \quad x \in (-1,1),
\]

and the end conditions,
\[
u(x, \tau) = s_1(x), \quad u_s(x, \tau) = s_2(x), \quad x \in (-1,1),
\]

and the boundary conditions,
\[
u(-1, \tau) = u_s(-1, \tau), \quad \nu(1, \tau) = u_s(1, \tau), \quad \tau \in (0, \tau),
\]

where \( \beta = \frac{2}{\ell} \). We approximate the equations (10)-(12) by

\[
u_s(x, t) = \beta^2 u_{ss}(x, t), \quad t \in (0, \tau),
\]

\[
u(x, 0) = u(x), \quad u_s(x, 0) = u_s(x),
\]

\[
u(x, \tau) = s_1(x), \quad u_s(x, \tau) = s_2(x), \quad k = 1, 2, ..., M - 1
\]

Substituting the equations (8) and (9) into the equations (14)-(16), we get a system of linear second order differential equations as

\[
\dot{a}_k(t) = \sum_{j=0}^{M} \beta^2 d_{kj}^{(2)} a_j(t), \quad t \in (0, \tau),
\]

\[
a_k(0) = u_1(x_k), \quad \dot{a}_k(0) = u_2(x_k),
\]

\[
a_k(\tau) = s_1(x_k), \quad \dot{a}_k(\tau) = s_2(x_k),
\]

\[
k = 1, 2, ..., M - 1.
\]

Moreover, from the equation (13) \( \alpha_k(\tau) = u_1(\tau), \quad \alpha_M(\tau) = u_2(\tau) \) we have; therefore, the equations (17)-(19) can be written in the matrix form as

\[
\dot{a}(t) = Ca(t) + B v(t),
\]

\[
a(0) = a_i, \quad \dot{a}(0) = a_i',
\]

\[
\alpha(\tau) = a_f, \quad \dot{a}(\tau) = a_f',
\]

where,

\[
C = \beta^2 \begin{bmatrix}
\alpha_{11}^{(2)} & \ldots & \alpha_{1M-1}^{(2)} \\
\ldots & \ldots & \ldots \\
\alpha_{M-11}^{(2)} & \ldots & \alpha_{M-1M-1}^{(2)}
\end{bmatrix},
\]

\[
B = \beta^2 \begin{bmatrix}
d_{ij}^{(2)} & \ldots & d_{ij}^{(2)} \\
\ldots & \ldots & \ldots \\
d_{iM-2}^{(2)} & \ldots & d_{iM-1}^{(2)}
\end{bmatrix}.
\]

and

\[
a(t)^T = (a_1(t), ..., a_{M-1}(t))^T, \quad v(t)^T = (v_1(t), ..., v_{M-1}(t))^T, \quad a_i = (u_1(x_i), ..., u_{M-1}(x_i))^T, \quad a_f = (s_1(x_i), ..., s_{M-1}(x_i))^T.
\]

Setting the control system (20)-(22) can be written as the following first order linear control system

\[
Y(t) = E Y(t) + F v(t),
\]

\[
Y(0) = \begin{bmatrix} a_i' \\ a_i \end{bmatrix}, \quad Y(\tau) = \begin{bmatrix} a_f' \\ a_f \end{bmatrix}
\]

where,

\[
Y(t) = \begin{bmatrix} \dot{a}(t) \\ a(t) \end{bmatrix}, \quad E = \begin{bmatrix} 0 & C \\ F & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} B \\ 0 \end{bmatrix}
\]

**Minimum energy Problem**

As a minimum energy problem (MEP), we first consider the objective functional (6) with \( p = 2 \). In
order to obtain the necessary conditions for the optimal controls \( v^*(t) = (v_1^*(t), v_2^*(t)) \) and the optimal time \( \tau^* \) we define the Hamiltonian as

\[
H(a, \dot{b}, u_1, u_2, \lambda_1, \lambda_2) = u_1^2 + u_2^2 + \lambda_1(Ca + Bv) + \lambda_2 \dot{b}.
\]

(25)

According to the Pontryagin's maximum principle (Kirk, 1970), we should have

\[
H(a^*, \dot{b}^*, u_1, u_2, \lambda_1^*, \lambda_2^*) \leq H(a^*, \dot{b}^*, u_1, u_2, \lambda_1, \lambda_2),
\]

(26)

for all control functions \( u_1 \) and \( u_2 \) satisfying the constraint (5), where \( \lambda_1^* \) and \( \lambda_2^* \) are the costate variables which satisfy

\[
\dot{\lambda}_1 = -\frac{\partial H}{\partial b} = -\dot{\lambda}_2,
\]

(27)

\[
\dot{\lambda}_2 = -\frac{\partial H}{\partial a} = -\lambda_2 \mathbf{C}
\]

(28)

The partial derivatives of the Hamiltonian with respect to \( b_j, j = 1,2 \) are zero. Since \( \frac{\partial H}{\partial v_j} = 2v_j \)

\[ -\dot{\lambda}_1 \mathbf{B}_j = 0, \]

we have \( v_j(t) = -\frac{1}{2} \dot{\lambda}_1^*(t) \mathbf{B}_j \) where \( \mathbf{B}_j, j = 1,2 \) the jth column of \( \mathbf{B} \). is to determine an explicit expression for the optimal controls, we consider the following three cases. (i) If

\[ \kappa_j \leq -\frac{1}{2} \dot{\lambda}_1^*(t) \mathbf{B}_j \leq \sigma_j, \]

then

\[ v_j^*(t) = -\frac{1}{2} \dot{\lambda}_1^*(t) \mathbf{B}_j. \]

(ii) If \( -\frac{1}{2} \dot{\lambda}_1^*(t) \mathbf{B}_j > \sigma_j, \) then

\[ v_j^*(t) = \sigma_j. \]

(iii) If \( -\frac{1}{2} \dot{\lambda}_1^*(t) \mathbf{B}_j < \kappa_j, \) then

\[ v_j^*(t) = \kappa_j. \]

Combining these three cases gives

\[
v_j^*(t) = \max\{\kappa_j, \min\{\sigma_j, -\frac{1}{2} \dot{\lambda}_1^*(t) \mathbf{B}_j\}\}, j = 1, 2.
\]

(29)

**Minimum fuel Problem**

Setting \( p = 1 \) in functional (6), leads to the minimum fuel problem (MFP). Assume that the bounds on the control functions in (5) are such that \( \kappa_j < 0 \) and \( \sigma_j > 0 \). The Hamiltonian function for this problem is

\[
H(a, \dot{b}, u_1, u_2, \lambda_1, \lambda_2) = \ |u_1| + |u_2| + \lambda_1(Ca + Bv) + \lambda_2 \dot{b}.
\]

(30)

According to the Pontryagin's maximum principle, the optimal controls \( v^*(t) = (v_1^*(t), v_2^*(t)) \) and the corresponding states \( b^*(t) \) and \( a^*(t) \) satisfy

\[
H(a^*, \dot{b}^*, u_1, u_2, \lambda_1^*, \lambda_2^*) \leq H(a^*, \dot{b}^*, u_1, u_2, \lambda_1^*, \lambda_2^*),
\]

(31)

for all control functions \( u_1 \) and \( u_2 \) satisfying the constraint (5), where \( \lambda_1^* \) and \( \lambda_2^* \) are the costate variables which satisfy the equations (27)-(28). From (31), it is concluded that

\[
v_j^*(t) = \begin{cases} \sigma_j & \text{if } -\lambda_1^*(t) \mathbf{B}_j \leq -1 \\ 0 & \text{if } -1 < -\lambda_1^*(t) \mathbf{B}_j < 1 \\ \kappa_j & \text{if } -\lambda_1^*(t) \mathbf{B}_j \geq 1 \end{cases}.
\]

(32)

Moreover, since \( H \) is explicitly independent of \( t \) and the final time \( \tau \) is free, we also know that (See Kirk (1970) for more details)

\[
|v_1^*(t)| + |v_2^*(t)| + \lambda_1^*(t)(Ca^*(t) + Bv^*(t)) + \lambda_2^*(t)b^*(t) = 0, \ t \in [0, \tau].
\]

(33)

According to the equation (32), a singular interval to exist it is necessary that \( \lambda_1^*(t) \mathbf{B}_j \) is to be either +1 or -1 during a time interval \([t_1, t_2]\). This implies that \( \lambda_1^*(t) \mathbf{B}_j = 0, \ t \in [t_1, t_2] \). Hence, from the equation (27) we have \( \lambda_1^*(t) \mathbf{B}_j = 0, \ t \in [t_1, t_2] \). Clearly, this condition occurs if \( \lambda_1^*(t) = 0 \) for \( t \in [t_1, t_2] \), but, this cannot happen, because from the equations (27)-(28) we obtain \( \lambda_1^*(t) = \lambda_1^*(t) \mathbf{B}_j = 0 \), \( t \in [0, \tau] \); hence, from the equation (33) we obtain

\[
v_j^*(t) = v_j^*(t) = 0, \ t \in [0, \tau],
\]

which means that the optimal control functions do not affect the system at all. We have a similar situation when \( B_j = 0 \). So, we consider the case that \( \lambda_1^*(t) \) is nonzero for any \( t \in [t_1, t_2] \). From the equations (27)-(28) we have \( \lambda_1^*(t) = -\lambda_2^* \mathbf{C}_j \), and hence twice differentiating the equation \( \lambda_1^*(t) \mathbf{B}_j = 0 \) gives \( \lambda_1^*(t) \mathbf{C}_j = 0 \). Similarly, twice differentiating the equation \( \lambda_1^*(t) \mathbf{B}_j = 0 \) gives \( \lambda_1^*(t) \mathbf{C}^2 \mathbf{B}_j = 0 \). Continuing this pattern gives

\[
\lambda_2^*(t) [\mathbf{B}_j | \mathbf{C}^2 \mathbf{B}_j | ... | \mathbf{C}^M \mathbf{B}_j] = 0, \ t \in [t_1, t_2].
\]

(34)

Obviously, if the equation (34) is to be satisfied, the matrix
must be singular. Therefore, if the matrices
\[ [B_j | CB_j | \cdots | C^{M-1}B_j] \]
\( j=1, 2 \) are nonsingular, then there are no singular intervals and the optimal controls are completely determined by (32).

We note that the optimality systems of the MEP and the MFP contain the state system (23) with the boundary conditions (24), the costate system (27)-(28), together with the expressions (29) and (32) for the control functions and a complicated nonlinear equation for the optimal time \( \tau^* \) as
\[ H (\alpha^* (\tau^*), \beta^* (\tau^*), \nu_1^* (\tau^*), \nu_2^* (\tau^*), \nu_3^* (\tau^*), \nu_4^* (\tau^*)) = 0, \]
(36)

where, \( H \) is the corresponding Hamiltonian function. Due to difficulties in solving the optimality systems of the MEP and the MFP, in the next section we directly use the control parameterization enhancing technique (CPET) to optimize the functional (6) subject to the constraints (5) and (23)-(24). The CPET introduced by Lee et al. (1997) maps all the switching points of the original problem onto the set of integers, so that the time of the switching points can be accurately determined.

**CONTROL PARAMETRIZATION ENHANCING TECHNIQUE**

Using the CPET, an optimal control problem can be approximated by an optimal parameter selection problem which can be solved efficiently by the software package MISTER3 (Jennings et al., 1997).

**CPET1 for MEP**

In order solve the MEP we set \( \mathcal{U} = \{ \kappa_1, \sigma_1 \} \times \{ \kappa_2, \sigma_2 \} \) and we assume that the control function \( \nu \) has a piecewise constant form as:
\[ \nu^N (t) = \sum_{i=1}^{N} \alpha_i \chi_{(t_{i-1}, t_i)} (t) \]
(37)

where \( N \) is the number of control subintervals and \( \chi_{(t_{i-1}, t_i)} \) is the characteristic function for the interval \( t_{i-1}, t_i \). We note that \( \alpha_i \in \mathcal{U} \) and \( t_{i}, k = 1, 2, \ldots, N \) the decision variables characterizing \( \nu \) where, \( 0 = t_{0} \leq t_{1} \leq \cdots \leq t_{N} = \tau \). Substituting the function (37) into (6) and (23)-(24) gives the following optimal control problem:

Minimize \[ J (\nu^N) = \sum_{i=1}^{N} \alpha_i \int_{t_{i-1}}^{t_{i}} \nu_2 (\tau) \, d\tau \]
(38)

Subject to
\[ Y (t) = E Y (t) + F \nu, \quad \nu, \, t \in [0, t_{N}]. \]
(39)

\[ Y (0) = \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right], \quad Y (t) = \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right], \quad k = 1, \ldots, N. \]
(40)

It has been proved that \( \lim_{N \to \infty} J (\nu^N) = J (\nu^* , \nu^*) \) where \( \nu^* \) is the optimal solution of (38)-(40) and \( (\nu^* , \nu^*) \) is the optimal solution of MEP satisfying the optimality conditions (29) and (36) (See chapter 6 of Teo et al. (1999)). Now, we relate the new time variable \( t \in [0, \tau) \) to the original time variable \( \tau \) through the ordinary differential equation
\[ \frac{d\tau}{ds} = \varrho (s), \quad \tau (0) = 0, \]
(41)

where \( \varrho : [0, \tau) \to \mathbb{R} \) is a new nonnegative piecewise constant function defined by
\[ \varrho (s) = \sum_{i=1}^{N} \beta_i \chi_{[t_{i-1}, t_i)} (s) \]
(42)

with \( \beta_i = t_i - t_{i-1} \); hence, \( t_N = \sum_{i=1}^{N} \beta_i, k = 1, \ldots, M. \)

Let \( Y (s) = Y (\tau (s)) \). Under the CPET transformation, the problem (38)-(40) becomes:

Minimize \[ J (\nu^N) = \sum_{i=1}^{N} \beta_i \alpha_i^2 \]
(43)

Subject to
\[ \frac{d\varrho (s)}{ds} = \beta_i (E Y (s) + F \alpha_i), \quad s \in [t_{i-1}, t_i], \]
(44)

\[ \varrho (0) = \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right], \quad \varrho (t) = \left[ \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right], \quad \alpha_i \geq 0, \quad k = 1, \ldots, N. \]
(45)

**CPET2 for MFP**

For the MFP, we use the CPET for optimal discrete-valued control problems introduced by Lee et al. (1999). To this end, we assume that the con-
control variable can take values in \( \mathcal{U} = \{ \{\alpha_j\} \}_j \). According to the necessary conditions for the nonsingular optimal controls given by (32), we have \( \mathcal{U} = \{ \{\alpha_j\} \}_j \). Hence, \( m = 9 \).

In this technique, we construct a set \( \mathcal{N} = \{ \alpha_j \}_j, N = nm \) where \( \alpha_1, \alpha_2, \ldots, \alpha_m = \alpha_1, \alpha_2, \ldots, \alpha_m = \alpha_1, \ldots, \alpha_m = \alpha_1 \), and we seek a control function in the form of (37). Therefore, in this case, \( t_j, j = 1, 2, \ldots, N \) is now determined only by \( t_j, j = 1, 2, \ldots, N \).

Moreover, we use the time scale control (42) to relate the new time variable \( s \in [0, M] \) to the original time variable through the ordinary differential equation (41) where \( \beta_k = t_k - t_{k-1} \). Following a procedure similar to CPET1, we obtain an optimal parameter selection problem as

Minimize \( J = \sum_{k=1}^{N} \beta_k \alpha_k \) \hspace{1cm} (46)

Subject to

\[
\begin{align*}
\frac{d\tilde{Y}_k^{*}(t)}{dt} & = \beta_k (E \tilde{Y}_k^{*}(t) + F \alpha_k), \quad t \in [k-1, k), \quad (47) \\
\tilde{Y}_0(0) & = \begin{bmatrix} d_1 \\ \alpha_1 \end{bmatrix}, \quad \tilde{Y}_N(N) = \begin{bmatrix} d_f \\ \alpha_f \end{bmatrix}, \quad \beta_k \geq 0, \quad k = 1, \ldots, N \quad (48)
\end{align*}
\]

Note that the unknowns in problem (46)-(48) are only \( \beta_k, k = 1, \ldots, N \), while in problem (38)-(40) \( \alpha_k, k = 1, \ldots, N \) are also unknowns. Obviously, the CPETs introduced in this paper lead to the optimal parameter selection problems, which can be solved by the optimal control software Miser 3.

**NUMERICAL RESULTS**

As a numerical example, we consider a problem with \( \ell = 100, \alpha = 1 \), \( u_1(z) = \frac{4}{\ell^2} (z^2 - \frac{\ell^2}{4}) \), \( u_2(Z) = 0 \), \( s_1(z) = \sin \left( \frac{2\pi z}{\ell} \right) \), \( s_2(Z) = 0 \), \( k_i = 1, \alpha_i = 1 \), \( j = 1, 2 \).

In our implementation we set \( M = 10 \). Fig. 1 and Fig. 2, respectively, show the optimal boundaries \( \nu^* \), \( j = 1, 2 \) and the corresponding optimal state \( u^*(x,t) \) for the MEP obtained by CPET1 with \( N = 10 \). The optimal value for objective functional (6) obtained by CPET1 is \( J \approx 23.3801 \) which is corresponding to \( \tau^* = 147.1297 \). The optimal controls obtained by the CPET2 with \( n = 5 \) and the corresponding optimal state \( u^2(x,t) \) are depicted in Fig. 3 and Fig. 4, respectively. Moreover, the value of objective functional (6) resulting by CPET2 is \( J \approx 58.1754 \) which is corresponding to \( \tau^* = 50.8477 \).

![Fig. 1. Control functions \( v_1(-) \) and \( v_2(-) \) for MEP obtained by CPET1.](image1.png)

![Fig. 2. \( u^2(x,t) \) for MEP obtained by CPET1.](image2.png)
CONCLUSION

In this paper, we approximated the boundary optimal control problem of the wave equation by a linear optimal control problem using a spectral method. When the objective functions indicate the minimum energy problem and the minimum fuel problem, the necessary optimality conditions are obtained. Moreover, the control parameterization enhancing technique is used to obtain the sub-optimal control functions in piecewise constant form. The numerical results confirmed the capability of the proposed method for solving the boundary optimal control problem of the one-dimensional wave equation.

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