



# A Method for Solving Convex Quadratic Programming Problems Based on Differential-Algebraic Equations

Masomeh Abbasi\*

Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran

**Received:** 22 February 2018

**Accepted:** 02 December 2018

**Keywords:**

Differential-algebraic equations  
Barrier function method  
Interior point method  
Convex quadratic programming  
Dynamic systems  
Stability

**Abstract**

In this paper, a new model based on differential-algebraic equations (DAEs) for solving convex quadratic programming (CQP) problems is proposed. It is proved that the new approach is guaranteed to generate optimal solutions for this class of optimization problems. This paper also shows that the conventional interior point methods for solving (CQP) problems can be viewed as a special case of the new DAEs methods. Numerical results show the efficiency of the proposed model.

\*Correspondence E-mail: [mas.abbasi81@gmail.com](mailto:mas.abbasi81@gmail.com)

**INTRODUCTION**

Differential-Algebraic Equations can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical systems, power systems, chemical process simulation, and optimal control. Many important mathematical models can be expressed in terms of DAEs (Riaza, 2008). In this paper a DAE system of index 1 is applied which is the following form

$$\begin{aligned} \dot{x} &= f(x,z,t) \\ g(x,z,t) &= 0 \end{aligned}$$

where  $x$  and  $z$  are called differential variables and algebraic variables respectively (Riaza, 2008). In this paper, a DAEs approach is used to solve convex quadratic problem. First, the logarithmic barrier function method is applied to combined the objective function and nonnegativity variables constrained into a composite function that called the barrier function. In the barrier function method, a family of problems in terms of parameter that usually indicate by  $\mu$  must solve. For a fixed  $\mu$ , KKT conditions of corresponding barrier problems are nonlinear algebraic equations that solutions of these equations equivalent to the corresponding barrier problems. Interior point methods can be applied to solve these algebraic equations that in these methods the most commonly used updating scheme for the barrier parameter  $\mu$  is (Potra & Wright, 2000)

$$\mu^{k+1} = (1-\alpha)\mu^k, \quad 0 < \alpha < 1$$

In this paper for updating barrier parameter, a differential equation is used, this differential equation coupled with algebraic equations provide a DAE. The rest of the paper is organized into four sections. In Section 2, differential-algebraic equations will be derived to solve convex quadratic programming problems. In Section 3, the convergence of the approach will be analyzed. In Section 4, illustrative examples will be given to demonstrate the performance of the proposed approach. Finally, Section 5 gives the conclusion of this paper.

**DIFFERENTIAL-ALGEBRAIC EQUATIONS PROBLEM FORMULATION**

Consider the following quadratic convex programming:

$$\begin{aligned} \text{Minimize} \quad & c^T x + 1/2 x^T A x \\ \text{subject} \quad & D x = b \\ & x \geq 0 \end{aligned} \tag{1}$$

where  $x \in \mathbb{R}^n$ ,  $D \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $c \in \mathbb{R}^n$ ,  $\text{rank}(D) = m \leq n$  and  $A \in \mathbb{R}^{n \times n}$  is symmetric positive semi definite.

The following problem ensue by the application of the logarithmic barrier function technique for the above problem (Fiacco & McCormick, 1990):

$$\begin{aligned} \text{Minimize} \quad & \phi(x, \mu) = c^T x + \frac{1}{2} x^T A x - \mu \sum_{j=1}^n \log x_j \\ \text{subject} \quad & D x = b \end{aligned} \tag{2}$$

where  $\mu > 0$  is the barrier penalty parameter.

**Theorem 2.1:** If feasible set of 1 (its dual) has a nonempty interior and is bounded, then for each  $\mu > 0$  there exists a unique solution  $(x_\mu, y_\mu, z_\mu)$  for barrier problem of 1.

**Proof:** see (Vanderbei, 2015)

For a fixed  $\mu$  the K.K.T conditions for 2 has the following parametric form (Bai et al., 2002; Bai et al., 2004):

$$\begin{aligned} D x &= b \\ D^T y - A^T x + z &= c \\ X Z e &= \mu e \end{aligned} \tag{3}$$

where  $X = \text{diag}(x_1, \dots, x_n)$ ,  $Z = \text{diag}(z_1, \dots, z_n)$ ,  $e = [1, \dots, 1]_{n \times 1}$

In (3), The first equation is the equality constraint that appears in 1, while the second equation is the equality constraint for the dual of 1 (Bazaraa et al., 2013). Furthermore, component-wise of the third equation as follows:

$$x_j z_j = \mu \quad j = 1, 2, \dots, n$$

As can be seen,  $x_j z_j = \mu$  ( $j = 1, 2, \dots, n$ ) are closely related to complementarity. In fact, if  $\mu = 0$ , then they are exactly the usual complementarity conditions that must be satisfied at K.K.T optimality

conditions. For this reason, these last equations called the  $\mu$ - complementarity conditions. So if  $(x^*, y^*, z^*)$  satisfy in two first equations of 3 and

$$x_j^* \cdot z_j^* = 0 \quad j=1,2,\dots,n$$

then  $(x^*, y^*, z^*)$  satisfies in the kuhn-tucker conditions for 1 and therefor is optimal solution for 1. Define

$$\theta(\mu) = \inf \left\{ -\mu \alpha \sum_{j=1}^n (\log x_j + \log z_j) \mid \right.$$

$$\left. Dx = b, D^T y - A^T x + z = c, X.Z = 0 \right\} \quad (4)$$

which is a convex program because both the objective function and the constraints are convex. By theorem 2.1 for any fixed  $\mu > 0$ ,  $\theta(\mu)$  has a unique solution and hence,  $\theta(\mu)$  is differentiable with respect to  $\mu$ . The derivative of the function is found as follows:

$$\frac{d\theta(\mu)}{d\mu} = -\alpha \sum_{j=1}^n (\log x_j + \log z_j) \quad (5)$$

It is well known from classical optimization theory (Bazaraa et al., 2013) that:

$$\inf_{\mu > 0} \theta(\mu) = \inf \{ 0 \mid Dx = b, D^T y - A^T x + z = c, X.Z = 0, x \geq 0, z \geq 0 \} \quad (6)$$

i.e., the optimal solution to 1 can be obtained by minimizing  $\theta(\mu)$ . Using the steepest-descent method, the following differential equation for minimizing  $\theta(\mu)$  obtained:

$$\frac{d\mu}{dt} = -\frac{d\theta}{d\mu} = \alpha \sum_{j=1}^n (\log x_j + \sum_{i=1}^n \log z_j) \quad (7)$$

where  $x = [x_1, x_2, \dots, x_n]$  and  $z = [z_1, z_2, \dots, z_n]$  satisfy 3.

In the rest of paper, let

$$S = \{ (x, y, z) \in R^n \times R^m \times R^n : Dx = b, D^T y + Z = c, x > 0, z > 0 \}$$

is not empty, if  $(x_0, y_0, z_0) \in S$  to be start point in this approach then for each  $\mu > 0$ , the parameterized system 3 has a unique solution. This solution is denoted as  $(x(\mu), y(\mu), z(\mu))$  and is called  $\mu$ -center. The set of  $\mu$ -centers (with  $\mu$  running through all positive real numbers) gives a homotopy path, which is called the central path. If  $\mu \rightarrow 0$  then the limit of the central path exists and since the limit points satisfy in , the limit yields optimal solutions for 1 (Bai et al., 2002; Bai et al., 2004)

The differential equation 7 and the algebraic equations 3 form the DAE for 1. The algebraic equations 3 can be transformed into differential equations as follows

$$D \frac{dx}{dt} = 0 \quad (8)$$

$$D^T y \frac{dx}{dt} - A^T \frac{dx}{dt} + \frac{dz}{dt} = 0 \quad (9)$$

$$Z \frac{dx}{dt} + X \frac{dz}{dt} = \frac{d\mu}{dt} e \quad (10)$$

Taking initial values for  $x, y, z$  satisfying the algebraic equations 3, and hence in the interior of the feasible region, and taking a small initial value for  $\mu$  hence, the differential equations 7 and 8-10 can be solved. In Section 3, it will be shown that the equilibria of these differential equations are the optimal solutions to 1.

### CONVERGENCE

In this section, the convergence of the trajectories of the DAE to solve 1 will be discussed

**Theorem 3.1:** Let  $(x_0, y_0, z_0) \in S$  and  $\mu_0$ . Suppose  $x(t), y(t), z(t), \mu(t)$  that denote the trajectories of the solution of 7 and 8-10 with initial values  $(x_0, y_0, z_0)$ . Then, either  $(x(t), y(t), z(t)) \in S$ , and  $\mu \geq 0$ , or  $x(t)$  is an optimal solution to 1.

**Proof:** First, it will be shown that  $x, y, z$  determined by equations 3, are continuous functions of  $\mu$ . Let

$$F_1(x, y, z, \mu) = XZ e - \mu e,$$

$$F_2(x, y, z, \mu) = Dx - b$$

$$F_3(x, y, z, \mu) = D^T y - A^T x + z - c$$

$$F = [F_1(x, y, z, \mu), F_2(x, y, z, \mu), F_3(x, y, z, \mu)].$$

then equations (3) can be rewritten as

$$F(x,y,z)=0. \tag{11}$$

Let  $J(x, y, z)$  be the Jacobian of  $F(x, y, z)$ . To apply the implicit function theorem (Apostol,1974; Xiong, 2002) to equation 11, calculation of the determinant of  $J$  is needed. Clearly,

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= Z, & \frac{\partial F_1}{\partial y} &= 0, & \frac{\partial F_1}{\partial z} &= X \\ \frac{\partial F_2}{\partial x} &= D, & \frac{\partial F_2}{\partial y} &= 0, & \frac{\partial F_2}{\partial z} &= 0 \\ \frac{\partial F_3}{\partial x} &= -A^T, & \frac{\partial F_3}{\partial y} &= D^T, & \frac{\partial F_3}{\partial z} &= I \end{aligned}$$

thus

$$|J(x_0,y_0,z_0)| = \begin{vmatrix} Z_0 & 0 & X_0 \\ D & 0 & 0 \\ -A^T & D^T & I \end{vmatrix} = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = |B|,$$

where

$$\begin{aligned} B_{11} &= Z_0, & B_{12} &= \begin{bmatrix} 0 & X_0 \end{bmatrix} \\ B_{21} &= \begin{bmatrix} D \\ -A^T \end{bmatrix}, & B_{22} &= \begin{bmatrix} 0 & 0 \\ D^T & I \end{bmatrix} \end{aligned}$$

But

$$|B| = |B_{11}| \cdot |B_{22} - B_{11}^{-1}B_{12}| \tag{12}$$

Clearly

$$\begin{aligned} B_{21}B_{11}^{-1}B_{12} &= \begin{bmatrix} D \\ -A^T \end{bmatrix} Z_0^{-1} \begin{bmatrix} 0 & X_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & DZ_0^{-1}X_0 \\ 0 & -A^T Z_0^{-1}X_0 \end{bmatrix} \\ B_{22} - B_{21}B_{11}^{-1}B_{12} &= \begin{bmatrix} 0 & -DZ_0^{-1}X_0 \\ D^T & I + A^T Z_0^{-1}X_0 \end{bmatrix} \end{aligned}$$

Thus

$$|B_{22} - B_{21}B_{11}^{-1}B_{12}| = |DZ_0^{-1}X_0D^T|. \tag{13}$$

by Combining (12) and (13) we have

$$|J(x_0,y_0,z_0)| = |Z_0| |DZ_0^{-1}X_0D^T|$$

Because  $Z_0^{-1}=1/\mu_0$ ,  $X_0, \mu_0 > 0$ ,  $x_0 > 0$ ,  $z_0 > 0$ , and since  $D$  has full row rank,  $|DD^T| \neq 0$ , we have

$$|DZ_0^{-1}X_0D^T| = \frac{1}{\mu_0} |X_0^2| |DD^T| \neq 0$$

So

$$|J(x_0,y_0,z_0)| \neq 0$$

The implicit function theorem implies that there are unique, differentiable, and vector-valued functions  $x(\mu)$ ,  $y(\mu)$ ,  $z(\mu)$  passing through  $x_0 = x(\mu_0)$ ,  $y_0 = y(\mu_0)$ ,  $z_0 = z(\mu_0)$ . Since the functions  $\sum_{j=1}^n \log x_j$  and  $\sum_{i=1}^n \log z_i$  are continuous in  $x$  and  $z$  respectively, and the vector-valued functions  $x$  are continuous in  $\mu$ , according to the theorem of existence and uniqueness of solutions of nonlinear differentiable equations, the function  $\mu$  is continuous in  $t$ . Therefore, the vector-valued functions  $x, y, z$  are also continuous in  $t$ . Now, it is shown that the solutions of the system of equations 8-10 satisfy equations 3. By integrating both sides of equation 8, we obtain:

$$\int_0^t D\left(\frac{dx}{dt}\right) d\tau = 0 \tag{14}$$

It follows from equation (14) that

$$D(x(t)) - D(x(0)) = 0$$

But

$$D(x(0)) = b$$

Thus

$$Dx(t) = b$$

Similarly,

$$\begin{aligned} D^T y(t) - A^T x(t) + z(t) &= c \\ X(t)Z(t)e &= \mu(t)e \end{aligned}$$

We now show that the trajectory of the solution is either in the interior of the feasible region or at the optimal solution. Because  $\mu_0 > 0$  and  $\mu(t)$  is continuous in  $t$ , before  $\mu(t)$  becomes negative, it

must be  $\mu(t^*)=0$  at some time  $t^*$ , which implies that

$$x_j(t^*)z_j(t^*)=0, \quad j=1,2,\dots,n \quad (15)$$

Previously, it was shown that  $x(t^*), y(t^*), z(t^*)$  satisfy (3). Equations (3) and (15) are the optimality conditions for (1). This shows that  $x_j(t^*)(j=1,2,\dots,n)$  are the optimal solutions.

Because  $\mu(t)=x_j(t^*)z_j(t^*)>0$  implies that either  $z_j(t)>0$  and  $x_j(t)>0$  or  $z_j(t)<0$  and  $x_j(t)<0$ .  $z_j(t)<0$  and  $x_j(t)<0$  cannot take place, Otherwise, because  $z_j(0)>0, x_j(0)>0$  and are continuous in  $t$ , before  $x_j(t)$  and  $z_j(t)$  become negative, there exists a time  $t^*$  such that  $x_j(t^*)z_j(t^*)=0(j=1,2,\dots,n)$ , which implies that  $x_j(t^*)$  are the optimal solution to 1.

Now, consider  $x_j(t)>0$  and  $z_j(t)>0$ . Previously, it was shown that

$$\begin{aligned} D^T y(t)-Ax(t)+z(t) &= c \\ Dx(t) &= b \end{aligned}$$

This means that the trajectory is in the set  $S$ . This completes the proof of Theorem 3.1.

We now show that the optimal solution to 1 is an asymptotically stable solution to the DAE.

**Theorem 3.2:** The dynamic system defined by the differential Eq. 7 and 8-10 is stable in the sense of lyapunov and is globally asymptotically convergent to the optimal solution of 1 . Moreover, the convergence rate of the DAE model increases as  $\alpha$  increases.

**Proof.** Let

$$V(\mu)=\theta(\mu)-\theta(\mu^*)$$

Where  $\mu^*$  minimizes  $\theta(\mu)$  for all  $\mu \geq 0$ . It is obvious that  $V(\mu)$  is positive definite. By Theorem 3.1, it was proven that the solutions  $\{x_j, j=1,2,\dots,n\}$  to the differential equations with initial values  $(x_0, y_0, z_0) \in S$  are the solutions to the system of nonlinear Eq. 3. Therefore, it follows from Eq. 5 and 7 that

$$\frac{dv}{dt} = \left[ \frac{d\theta(\mu)}{d\mu} \right] \frac{d\mu}{dt} = -\alpha \left[ \frac{d\theta(\mu)}{d\mu} \right]^2 < 0 \quad (16)$$

This shows that  $V(\mu)$  is a lyapunov function and stable points of the differential Eq.7 and 8-10 are the minimizers of  $\theta(\mu)$ , but

$$\inf_{\mu \geq 0} \theta(\mu) = \inf \left\{ c^T x + \frac{1}{2} x^T A x \mid Dx = b, x > 0 \right\} \quad (17)$$

So, the differential Eq. 7 and 8-10 is stable in the sense of lyapunov and is globally asymptotically convergent to the optimal solution of 1. The inequality in 16 implies that the faster the descent rate of  $V(\mu)$  is, the larger the  $\alpha$  is. Therefore, the convergence rate of DAE model increases as  $\alpha$  increases. The proof of Theorem 3.2 is complete.

### ILLUSTRATIVE EAMLES

In this section, it is demonstrated the behavior of the differential-algebraic approach for solving convex quadratic programming using illustrative examples. The simulation is conducted on Matlab, the ordinary differential equation solver engaged is ode15s.

**Example 1:** Consider the following convex quadratic program:

$$\begin{aligned} \text{Minimize} \quad & -4x_1 + x_1^2 - 2x_1 x_2 + 2x_2^2 \\ \text{subject} \quad & 2x_1 + x_2 \leq 6 \& \\ & x_1 - 4x_2 \leq 0 \& \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

The exact solution of this problem is (32/13, 14/13). The initial values for  $x_0, y_0, z_0, \mu, \alpha$  are chosen as follows:

$$\begin{aligned} \mu_0 &= 0.4, \\ \alpha &= 10, \\ x_0 &= [1.8153, 0.5312, 1.8382, 0.3093], \\ y_0 &= [-2.2028, -0.1649], \\ z_0 &= [3.1388, 0.0374, 2.2028, 0.1649]. \end{aligned}$$

It is shown that proposed approach converges to following point which is the solution of the above problem.

$$x_1 = 2.4615, x_2 = 1.0769$$

The trajectories of the solution are shown in Figs. 1(a) and 1(b). The transient behavior of the differential-algebraic approach in terms of  $\mu$  is shown in Fig. 1(c). Let  $\alpha=100$ , Fig. 1(d) shows that under same initial points, the convergence rate of DAE model increases as  $\alpha$  increases.

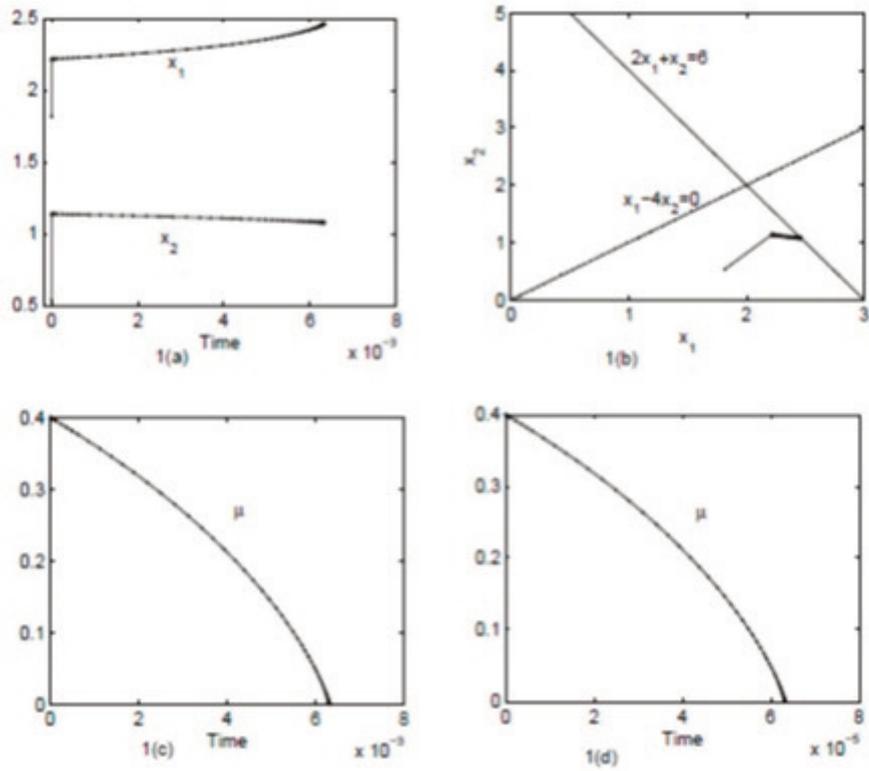


Fig.1. Transient behavior of the DAE model in example 1

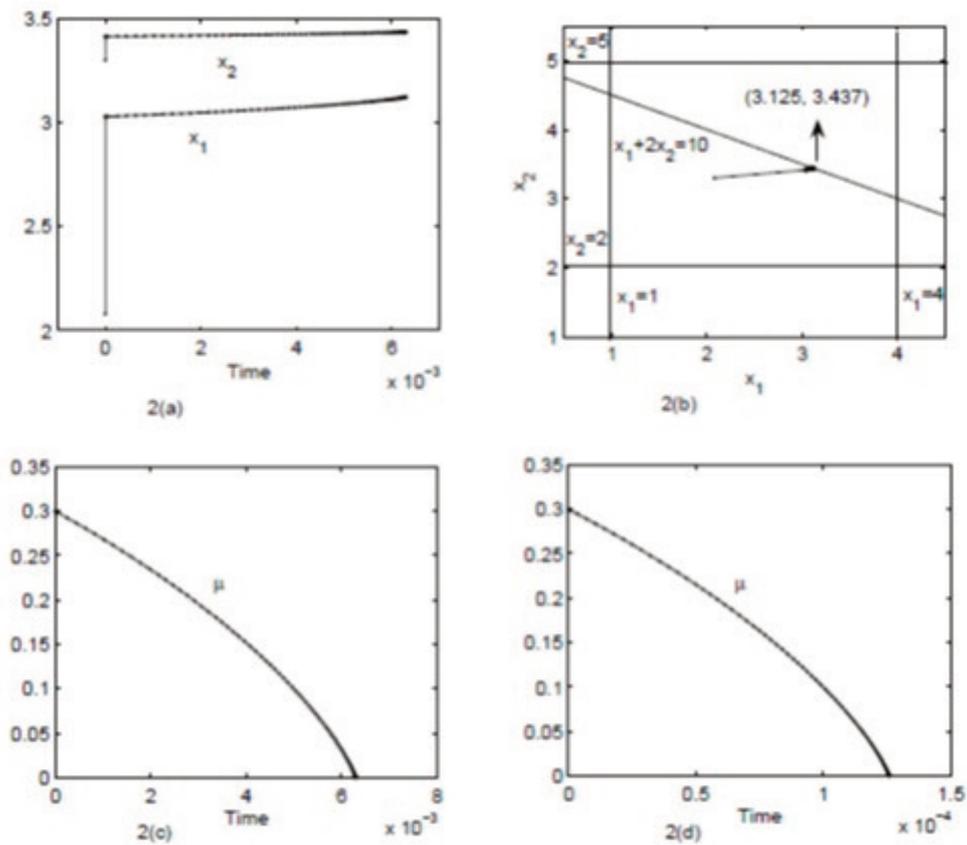


Fig.2. Transient behavior of the DAE model in example 2

**Example 2:** Consider the following bounded convex quadratic program:

$$\begin{aligned} & \text{Maximize} && 5x_1-x_1^2+x_1x_2-2x_2^2 \\ & \text{subject} && 2x_1+2x_2\leq 10 \\ & && 1\leq x_1\leq 4 \\ & && 2\leq x_2\leq 5 \end{aligned}$$

The exact solution of this problem is (3.125, 3.437). The initial values for  $x_0$ ,  $y_0$ ,  $z_0$ ,  $\mu$ ,  $\alpha$  are chosen as follows:

$$\begin{aligned} \mu_0 &= 0.3, \\ \alpha &= 5, \\ x_0 &= [1.0779, 1.3014, 1.3193, 1.9221, 1.6986], \\ y_0 &= [-2.0728, -2.3524, -0.2801], \\ z_0 &= [0.2797, 0.5533, 2.0728, 2.3524, 0.2801]. \end{aligned}$$

It is shown that proposed approach converges to following point which is the solution of the above problem.

$$x_1=3.125, x_2=3.437$$

The trajectories of the solution are shown in Figs. 2(a) and 2(b). The transient behavior of the differential-algebraic approach in terms of  $\alpha$  is shown in Fig. 2(c). Let  $\alpha=250$ , Fig. 2(d) shows that under same initial points, the convergence rate of DAE model increases as  $\alpha$  increases.

## CONCLUDING

In this paper, it was shown that a convex quadratic programming can be convert to a differential-algebraic model where the optimal solution of convex quadratic programming is the only equilibrium point of the DAE. Two illustrative examples have been given to demonstrate the functional capability and operational characteristics of the proposed model. These results show that the differential algebraic approach a promising alternative for solving convex quadratic programming problems.

## REFERENCE

Apostol, T. M. (1974). Mathematical analysis.  
Bai, Y. Q., El Ghami, M., & Roos, C. (2004). A comparative study of kernel functions for primal-dual interior-point algorithms in linear

optimization. *SIAM Journal on Optimization*, 15(1), 101-128.

Bai, Y. Q., Roos, C., & El Ghami, M. (2002). A primal-dual interior-point method for linear optimization based on a new proximity function. *Optimization Methods and Software*, 17(6), 985-1008.

Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (2013). *Nonlinear programming: theory and algorithms*. John Wiley & Sons.

Fiacco, A. V., & McCormick, G. P. (1990). *Nonlinear programming: sequential unconstrained minimization techniques* (Vol. 4). Siam.

Potra, F. A., & Wright, S. J. (2000). Interior-point methods. *Journal of Computational and Applied Mathematics*, 124(1-2), 281-302.

Riaza, R. (2008). *Differential-algebraic systems: analytical aspects and circuit applications*. World Scientific.

Vanderbei, R. J. (2015). *Linear programming*. Heidelberg: Springer.

Xiong, M., Wang, J., & Wang, P. (2002). Differential-algebraic approach to linear programming. *Journal of optimization theory and applications*, 114(2), 443-470.