Iranian Journal of Optimization
Volume 10, Issue 1, 2018, 19-29
Research Paper
Online version is available on: www.ijo.iaurasht.ac.ir

Islamic Azad University
Rasht Branch
E-ISSN:2008-5427

An Input-Oriented Radial Measure for Returns to Scale Aggregation

Reza Kazemi Matin[1*], Roza Aziz[1] and Mahdi Mirjaberi[2]

[1]Department of Mathematics, Islamic Azad University, Karaj Branch, Karaj, Iran
[2] Department of Mathematics, Islamic Azad University, Khorasgan Branch, Isfahan, Iran

Received: 05 July 2017
Accepted: 14 August 2017

Abstract

In production theory, it is necessary to be capable of predicting the production function's long-run behaviors. Hereof, returns to scale is a helpful concept. Returns to scale describes the reaction of a production function to the proportionally scaling all its input variables. In this regard, Data envelopment analysis (DEA) provides a com- prehensive framework for returns to scale evaluation. A sequence of attempts has been made on the subject of returns to scale in DEA literature which cause DEA to be expanded to widespread applications. Centralization of carried out studies in firm level, on one hand, and the importance of economical interoperation in performance analysis in industry level, on the other hand, were the main motivation to start a new range of studies around identifying the return to scale in industry level. This paper collaborates interesting relations between firms and industry technology with performance analysis techniques to extract a relation between returns to scale status of firms and system-wide unit based on the reference set method.

Keywords:
Data envelopment analysis (DEA)
Input-oriented radial measure
Target setting
Returns-to-scale (RTS)
Aggregation
Industry

***Correspondence E-mail**: rkmatin@gmail.com

INTRODUCTION

In Economics, the term “returns to scale” describes the behavior of production or returns when all productive factors are increased or decreased in the same ratio simulta- neously. When all inputs are changed in the same proportion, we call this as a change in scale of production. The way total output changes due to change in the scale of produc- tion is known as returns to scale. While change in output in the short-run is associated with the change in factor proportions, the change in output in the long-run is associated with change in the scale of production. Thus, returns to scale is the long-run concept.

Estimating returns to scale at firm level, as a firm/enterprise is considered regarding a set of homogenous units, has been investigated widely in literature (Cooer et al., 1996; Cooer et al., 2006; Banker et al., 1996; Zarepisheh et al., 2010) At industry level, as an aggregation of a number of homogenous (in input and output) firms, estimating returns to scale essentially depends on the efficiency analysis at both industry and firm levels. To address the relationship between the efficiency analysis at industry and firm levels, the technique of data envelopment analysis (DEA) is employed widely as a suitable non-parametric and mathematically based technique for three folds: technology, cost, and allocation of resources.

Farrell, (1957) proposed “structural efficiency” concept for the industry efficiency analysis. During last decades researchers, especially the economists, has tended to dig into the subject employing DEA framework. Johansen, (1972) in terms of technology, with focus on short-run technologies, tried to study the technical efficiency at industry level. Being follow-on, F¨are et al. (1992) employ Johansen, (1972) model in a single output and firm specific inputs. Li and Ng, (1995) argued that the measures of efficiency, introduced by Førsund and Hjalmarsson, (1979) are not compatible with aggregate problem efficiency analysis. In ad- dition, he addressed the aggregation of firm technologies problem and carried out some valuable relations between technologies at both firm and industry levels. These useful relations between technologies of firms and industry would cause some analogous prop- erties at both levels which lead to similarity economical inter-operation at those levels. F¨are and zelenyuk, (2005) provides a mathematically consistent and theoretically justified way of aggregation of Farrell-type efficiency scores. Leleu and Brice, (2009) by considering some assumption as Li and Ng, (1995) evaluated efficiency measure in aggregation problems without information on price data. Aparicio et al. (2013) introduced an overall measure of technical inefficiency at the industry level by modified directional distance function.

As mentioned before, the industry question has been an interesting issue for scientists and economists for the past half century. The importance of economic aspect in efficiency analysis on one side, and the absence of economical concept in DEA as a missing link on the other side, has lead to introducing the concept of returns to scale (RTS) within DEA framework, as an economical interpretation. Indeed, this concept, in turn, expanded the applicability of DEA.

In common economic definition, the concept of RTS is expressed the relation between a proportional change in inputs to a productive process and the corresponding propor- tional change in output. If the same percent change in input and output observed, constant returns to scale (CRS) prevails. If output increases by a larger percentage than inputs, increasing returns to scale (IRS) prevails. And, if the output rises by a lower percentage than inputs, decreasing returns to scale (DRS) prevails. Therefore, in microeconomic, returns to scale provides an instructive insight into the extension or restriction of a firm. So, it provides useful information on the optimal size of firms.

There are many contributions in DEA which discuss the theory and applications of re- turns to scale at firm level. Banker, (1981) in his thesis, employed the standard DEA models to analyze returns to scale concept before all else. Afterward, Banker et al. (1984), consid- ering the unique optimal solution for Banker, (1981) models, addressed the RTS question. Banker, (1984) introduced the concept of a most productive scale size (MPSS) enterprise, and argued that this concept exists when at least one output increasing at a rate that is less than proportionate to the rate at which all inputs are increased.

Banker and Thrall, (1992) revealed a characterization of increasing, constant and decreas- ing RTS based on the sign of multipliers of BCC optimal solutions. Also, they offered a measure of scale elasticity and provided a model for determining its bounds. F¨are et al. (1985) have developed FGL model to determine the RTS at an observation. Detailed discussions can be found in Cooper et al. (1996), and Cooper et al. (2006). Tone and Sahoo, (2006) proposed an approach for measuring quantitative estimates of returns to scale as scale elasticity based on cost efficiency model.

Although the concept of returns to scale is defined only at efficient sections of the production frontier, several studies extended this concept to inefficient firms by projecting them on the efficient frontiers. In this case, returns to scale depends on the projection method. Fare et al. (1985), alike Banker et al. (1996), addressed this question by employing a two step approach. They, in first step, solved the BCC (Banker et al., 1984) and CCR (Charnes at al.,1978) models. In second step, a linear program for each non constant returns to scale firm is solved. Therefore, they need to solve three LPs for non constant returns to scale firms. On the other hand, Banker et al. (1996) solved the CCR model in step one and then the other LP for each firm with non constant returns to scale characteristics in step two. Likewise, Tone, (1996) argued that the seminal work of Banker, (1981) did not consider the inefficient firms and only the efficient firms are intended. Therefore, if a large number of firms were inefficient then we could not characterize the RTS of majority of firms.

As mentioned before, vast ranges of studies around RTS at firm level have been formed. On the other hand, the importance of economical view of industry efficiency analysis is deniable .The current paper, discuses around the question of returns to scale at industry level. Indeed, based on the relations between firms and industry technologies, some interesting relations between returns to scale in both firm and industry level are presented.

The rest of this paper is organized as follows. In the next section, we give a brief review of the approach proposed by Li and Ng, (1995) and then, returns to scale and related works in DEA framework will be studied. The proposed approach on returns to scale of firm/industry is given in section three. In section four, the applicability of the proposed approach is illustrated with an example. Conclusions appear in the last section.

BACKGROUND

Li and Ng, (1995) by generalizing Farrell, (1957) and Førsund and Hjalmarsson, (1979) developed a framework for analyzing the industry efficiency. They considered a production group consists of firms and argued that the technology of group is related to assumption on firm's technologies. They considered a set of firms and $(x^k, y^k) \in R_+^{M+N}$ as the inputs, output vector of firm k, k = 1, . . . , K. Also, the technology which firm k belong to is introduced as Tk . According to the aggregate concept, they defined the industry technology as below:

$$T^I = \left\{ (X,Y); (X,Y) = \sum_{k=1}^{K} (x^k, y^k), (x^k, y^k) \in T^k, k = 1,\dots,K \right\} \quad (1)$$

Li and Ng, (1995) with some assumptions on firm's technology including the convexity and identity ofirm technologies, denoted by T, showed that:

$$T^I = \sum_{k=1}^{K} T^k = \sum_{k=1}^{K} T = KT \quad (2)$$

Also, they stated that the firm technology is a convex cone (constant returns to scale technology) if and only if

$$T^I = KT = T \quad (3)$$

Taking the advantage of the mentioned links between industry and firm technologies, they introduced three measures of technical, allocative and overall industry efficiencies. Also, these useful relations between firm and industry technologies yield to some anal- ogous properties in both levels which turns into similarly economical inter-operation in both levels.

RTS in DEA models

In efficiency analysis, beside developing computational aspects, usually in DEA framework as a nonparametric approach, it is always essential to consider economic aspects. Returns to scale is probably the major motivation in a wide range of studies during last decades.

$$T_{CRS}=\{(x,y);x\geq X\lambda,y\leq Y\lambda,\lambda\geq 0\} \quad (4)$$

where $(X,Y)\in R^{M+N}\times R^{S+K}$ denote the firms input and output matrices and $\lambda\in R_{+}^{k}$. The input oriented CCR model is formulated, by Charnes et al. (1978), as follows:

$$\begin{aligned} &\text{Minimize} && \theta-\varepsilon(s^{-}+s^{+}) \\ &\text{Subject to} && \theta x_o - X\lambda - s^{-}=0 \\ &&& Y\lambda - s^{+}=y_o \\ &&& \lambda\geq 0, s^{-}\geq 0, s^{+}\geq 0 \end{aligned} \quad (5)$$

where s^{-}, s^{+} represent input excess and output shortfall, respectively. Here we have combined the two steps in a single model as it is usual in the literature. Let DMUo denote the firm under evaluation. Then the peer set for this firm is given by:

$$E_C^O=\{k,\lambda_k^{*}>0, k=1,\dots,K\} \quad (6)$$

The technology allowing for variable returns to scale (VRS) introduced by Banker et al. (1984) is then:

$$T_{VRS}=\{(x,y);x\geq X\lambda,y\leq Y\lambda,\lambda\geq 0,e\lambda=1\} \quad (7)$$

Also, the relevant DEA model, called BCC model in favor of Banker, Charnes, and Cooper, in envelopment form is given as:

$$\begin{aligned} &\text{Minimize} && \theta-\varepsilon(s^{-}+s^{+}) \\ &\text{Subject to} && \theta x_o - X\lambda - s^{-}=0 \\ &&& Y\lambda - s^{+}=y_o \\ &&& e\lambda=1 \\ &&& \lambda\geq 0, s^{-}\geq 0, s^{+}\geq 0 \end{aligned} \quad (8)$$

The multiplier (dual) form of the input oriented radial BCC model (without slacks) is given as:

$$\begin{aligned} &\text{Maximize} && z=uy_o-u_0 \\ &\text{Subject to} && -vX+uY-u_0\leq 0 \\ &&& vx_o=1 \\ &&& u\geq 0, v\geq 0 \end{aligned} \quad (9)$$

Similar to the CCR model, the BCC reference set $E^O{}_B$ is defined as:

$$E_B^O=\{k,\lambda_k^{*}>0, k=1,\dots,K\} \quad (10)$$

Now, we turn to the Banker and Thrall, (1992) approach in RTS evaluation.

Theorem 1. Assuming the observed firm is BCC-efficient and let the supremum and infimum optimal values of dual variable u0, corresponding to the convexity constraint in the BCC envelopment form, in the optimal solution of 9 are respectively denoted by $u^{-*}{}_0$ and $u^{+*}{}_0$ Then, we have;

1.if $u^{+*}{}_0 < 0$ then increasing returns to scale (IRS) is prevail in the firm.

2.if $u^{-*}{}_0 > 0$ then decreasing returns to scale (DRS) is prevail in the firm.

3.if $u^{-*}{}_0 \leq 0 \leq u^{+*}{}_0$ then constant returns to scale (CRS) is prevail in the firm.

Consequently, Tone, (1996) for BCC-efficient firms claimed that the firm is clearly CCR- efficient if and only if it is BCC-efficient and exhibits CRS. Also, he argued that in case of BCC-inefficiency, the reference set E_B , does not include both IRS and DRS firms. Accordingly:

Theorem 2. Let $E_B{}^o$ denote the reference set to a BCC-inefficient observed firm. Then, $E_B{}^o$ consists of BCC-efficient firms so that:

1.All the firms exhibit IRS;

2.The firms exhibit either IRS or CRS;

3.All the firms exhibit CRS;

4.The firms exhibit either CRS or DRS.

5.All the firms exhibit DRS.

Theorem 3. Let $(\hat{x}_o, \hat{y}_o)$ denotes the projection of an observed activity (x_o, y_o) onto the BCC-efficient frontier with the reference set EB. Then, $(\hat{x}_o, \hat{y}_o)$ exhibits:

1.IRS, if E_B consists of firms with either IRS or a mixture of IRS and CRS;

2.DRS, if E_B consists of firms with either DRS or a mixture of DRS and CRS.

Therefore, Tone(1996) has well characterized the CCR-efficient, BCC-efficient and BCC- inefficient firms in his work.

RETURNS TO SCALE:FIRMS VS. INDUSTRY

In this section, we want to determine the relevance between returns to scale of firms and returns to scale of an aggregated industry unit. In the literature, an industry unit is defined as the aggregation of all the constituent firms of that industry. As mentioned before, it is assumed that the industry is composed of K firms whose technology is defined as T^k, $k = 1, ..., K$.

Following Li and Ng, (1995) the firm technologies are admit to be identical, denoted by T , and satisfy the following regularity conditions:

1.“No free lunch is possible”: ”: $(0,y)\notin T$ for any $y\neq 0$;

2.“Doing nothing is possible”: $(x,0)\in T$ for any $x\in R_+^N$;

3.The set T is closed and convex;

4.The output set $P(x)=\{y:(x,y)\in T\}$ is bounded for any $x\in R_+^N$;

5.Free disposability of inputs and outputs: $(x_0, y_0)\in T$ implies that $(x,y)\in T$ for any $x\geq x_0$ and $y\leq y_0$;

6.Variable returns to scale: An observed firm in T may exhibit constant, increasing, or decreasing returns to scale.

Remark. Hereinafter in this paper, by the following notations, the corresponding concept is intended unless otherwise is understood from the context:

DMU_j : The observed firm j;

DMU_I: The aggregated industry unit;

DMU_j^I: The firms whose inputs and outputs are respectively K times of inputs and outputs of the observed firm j.

In production economics, returns to scale could be considered as a characteristic of the surface of the production technology. In this regard, it is possible to relate the returns to scale between technologies in industry and its constituent firms levels.

Theorem 4. If the firm technologies exhibit non-increasing returns to scale (NIRS), then the corresponding industry technology also exhibits NIRS.

Proof. Assuming the firm technologies satisfy NIRS, then for $\theta\geq 1$ we have $\theta T^k\subseteq T^k$ for $k=1, ..., K$. So, we can write $\theta T^I\subseteq\theta(KT)=K(\theta T)\subseteq KT$. According to (2.2) we have $\theta T^I\subseteq T^I$. This completes the proof.

Theorem 5. If the firm technologies satisfy non-decreasing returns to scale (NDRS), then the industry technology also exhibits NDRS.

Proof. The proof is omitted due to the similarity to theorem (4).

To study the relationship between returns to scale at firms and industry levels in a technology allowed for variable returns to scale, using DEA models, we need to investigate units on frontiers at both levels:

Theorem 6. The efficiency measure of DMU_O in T is identical to the efficiency measure of DMU_O^I in T^I.

Proof. Letting $\mathbf{X}o = \mathbf{K}\mathbf{x}o$ and $\mathbf{Y}o = \mathbf{K}\mathbf{y}o$ we have:

$$\min\{\theta;(\theta\mathbf{x}_o,\mathbf{y}_o)\in KT\}$$
$$=\min\{\theta;(\theta\mathbf{x}_o/K,\mathbf{y}_o/K)\in KT/K\} \quad (11)$$
$$=\min\{\theta;(\theta\mathbf{x}_o,\mathbf{y}_o)\in T\} \quad (12)$$

Theorem 6 simply states that, regardless of technology shape, if an observed firm is on the efficient frontier in T , its correspondence is also on the efficient frontier in T^I and vice versa. Now, the question that may come to mind is the relation between the projection of inefficient firms in both technology. Next theorem would answer this question:

Theorem 7. Supposing the observed firm (xo, yo) is inefficient. $(\mathbf{x}\hat{}_O, \mathbf{y}\hat{}_O)$ is the projection of DMUo on the efficient frontier in T , if and only if, $(\mathbf{Kx}\hat{}_O, \mathbf{Ky}\hat{}_O) = (\mathbf{X}\hat{}_O, \mathbf{Y}\hat{}_O)$ is the projection of $(\mathbf{Kx}_O, \mathbf{Ky}_O) = (\mathbf{X}_O, \mathbf{Y}_O)$ on the efficient frontier of T^I .

Proof. Let $(\theta_0^*, S^{-*}, S^{+*})$ is the optimal solution of model 8 and $(\mathbf{X}\hat{}_O, \mathbf{Y}\hat{}_O)$ is the projection of DMUI in T^I:

$$\begin{aligned}(\hat{X}_o,\hat{Y}_o)&=(\theta_o^*X_o-s^{-*},Y_o+s^{+*})\\&=K((\theta_o^*X_o-s^{-*})/K,(Y_o+s^{+*})/K) \quad (13)\\&=K((\theta_o^*x_o-\frac{s^{-*}}{K}),(y_o+\frac{s^{+*}}{K}))=(\hat{x}_o,\hat{y}_o)\end{aligned}$$

We know $(\mathbf{x}\hat{}_O, \mathbf{y}\hat{}_O)$ is the projection of DMUo on the efficient frontier of T , and this completes the proof.

In the following theorem we seek the relation of DMU_O and DMU_I in view of their RTS status.

Theorem 8. Returns-to-scale for an observed efficient of T is the same as its correspondence unit in T^I .

Proof. Banker and Thrall, (1992) and Banker et al. (1996) have proved that if ($\mathbf{x}_O$, $\mathbf{y}_O$) is on the efficient frontier of BCC model 8, the following conditions identify the returns to scale at this point with the sign of u_0^*

1.Increasing returns-to-scale prevails at ($\mathbf{x}_O$,$\mathbf{y}_O$) if and only if $u_0^* < 0$ for all optimal solutions.

2.Decreasing returns-to-scale prevails at ($\mathbf{x}_O$,$\mathbf{y}_O$) if and only if $u_0^* > 0$ for all optimal solutions.

3.Constant returns-to-scale prevails at (xo,yo) if and only if $u_0^* = 0$ in any optimal solution.

In BCC evaluation of DMU_O^I we have:

$$\begin{aligned} &\text{Maximize} \quad Z = uy_o - u_0 \\ &\text{Subject to} \\ &-vX_j + uY_j - u_0 \le 0, \quad j = 1,\dots,K \\ &vx_o = 1 \\ &u \ge 0, v \ge 0 \end{aligned} \tag{14}$$

where $(X_O, Y_O) = (K\boldsymbol{x}_O, K\boldsymbol{y}_O)$. Now, let $\boldsymbol{u}' = K\boldsymbol{u}$ and $\boldsymbol{v}' = K\boldsymbol{v}$, then 3.6 takes the following form:

$$\begin{aligned} &\text{Maximize} \quad Z = u'y_o - u_0 \\ &\text{Subject to} \\ &-v'X_j + u'Y_j - u_0 \le 0, \quad j = 1,\dots,K \\ &v'x_o = 1 \\ &u' \ge 0, v' \ge 0 \end{aligned} \tag{15}$$

It is clear that u_0^* is identical in value for both models 9 and 15. Therefore, both DMU_O and DMU_O^I exhibit CRS, $u_0^* = 0$ if and only if, in their corresponding model.

Now assuming $u_0^* > 0$ According to Banker et al. (1984), the following model should be solved for DMU_O to establish its returns to scale. In this model, if $u_0^{-*} = 0$, DMU_O exhibits constant returns to scale, otherwise $u_0^{-*} > 0$ and DMU_O exhibits decreasing returns to scale.

$$\begin{aligned} &u_0^{-*} = \text{Minimize} \quad u_0 \\ &\text{Subject to} \\ &-vx_j + uy_j - u_0 \le 0, \quad j = 1,\dots,K, j \ne 0 \\ &-v\hat{x}_o + u\hat{y}_o - u_0 = 0 \\ &v\hat{x}_o = 1, u\hat{y}_o - u_0 = 1 \\ &u \ge 0, v \ge 0, u_0 \ge 0 \end{aligned} \tag{16}$$

where ($\mathbf{x}\hat{}_O$, $\mathbf{y}\hat{}_O$) is the projection of ($\mathbf{x}_O$, $\mathbf{y}_O$) obtained from the envelopment form of model 8. Similarly, for DMU_O^I we have:

$$\begin{aligned} &u_0^{-*} = \text{Minimize} \quad u_0 \\ &\text{Subject to} \\ &-vX_j + uY_j - u_0 \le 0, \quad j = 1,\dots,K, j \ne 0 \\ &-v\hat{X}_o + u\hat{Y}_o - u_0 = 0 \\ &v\hat{X}_o = 1, u\hat{Y}_o - u_0 = 1 \\ &u \ge 0, v \ge 0, u_0 \ge 0 \end{aligned} \tag{17}$$

where ($\boldsymbol{X}\hat{}_O$, $\boldsymbol{Y}\hat{}_O$) is the projection of DMU_O^I . Again with substitutions $\boldsymbol{u}' = K\boldsymbol{u}$ and $\boldsymbol{v}' = K\boldsymbol{v}$, we have:

$$\begin{aligned} &u_0^{-*} = \text{Minimize} \quad u_0 \\ &\text{Subject to} \\ &-v'x_j + u'y_j - u_0 \le 0, \quad j = 1,\dots,K, j \ne 0 \\ &-v'\hat{x}_o + u'\hat{y}_o - u_0 = 0 \\ &v'\hat{x}_o = 1, u'\hat{y}_o - u_0 = 1 \\ &u' \ge 0, v' \ge 0, u_0 \ge 0 \end{aligned} \tag{18}$$

If u_0^{-*}=0 (>0) in model 16, then u_0^{-*}=0 (>0) in model 18, because the optimal solutions of both models are identical, and the only difference between them are the weights ($\boldsymbol{v}$,$\boldsymbol{u}$).

Now, assuming $u_0^* < 0$ according to Banker et al. (1984), it is sufficient to replace $u_0 \le 0$ with $u_0 \ge 0$ in model 16 and change its objective function from minimization to maximization and run the model. In the model, if the optimal value equals zero, u_0^{+*}=0 DMU_O exhibit constant returns to scale, otherwise $DMUo$ exhibits increasing returns to scale.

In a word, under the above arguments since returns to scale at a point is established by considering the sign of u_0, which remains unchanged, for both the observed firm and its correspondence in their technologies, the result is at hand.

Now, we are able to determine returns to scale status of an industry unit in DEA variable returns to scale technology, and show its relation with constitute firms returns to scale. To do so, first, we evaluate returns to scale of all efficient observed units DMU_j in T whose returns to scale are the same as their corresponding efficient units in the industry level. Then, we determine the reference set of $DMU\ I$ in T^1 to achieve the corresponding subset of efficient observed firm in T , with respect to the relevance of $DMU\ I$ and DMU_j . Then the returns to scale of the aggregated industry unit will be obtained as follows:

1.*DMUI* exhibits increasing return to scale, if it is constituted by increasing returns to scale or by mixture of increasing and constant return-to-scale firms of T .

2.*DMUI* exhibits decreasing return-to-scale, if it is constituted by decreasing returns to scale or by mixture of decreasing and constant returns to scale firms of T .

3.If all the constituted firms of *DMUI* exhibit constant returns to scale, the sign of u_0 should be determined as in theorem (8).

In the next section, we will illustrate our results by a numerical sample.

CASE STUDY

This section is intended to illustrate the major results of paper. Here, we devote our attention to banking industry in Iran. In terms of types, Iranian banks are categorized into three major divisions: Specialized banks, Commercial banks, Qardul-hassan (Non- Interest) banks. In terms of ownership, these banks also classified into three divisions: private banks, semipublic banks and state banks.

Regardless of operating scope and ownership, these banks provide their customers with services like accounts opening, debit card issuing, lending, and guarantees issuing. Insurance services are also provided by their subsidiaries. All Iranian banks should comply their entire operations with the rules and regulations of Islamic banking and banking operations without usury. These rules are drafted by the Supreme Council of Money and Credit, approved by the Jurisprudence Committee, notified and monitored by the Central Bank of Iran (CBI).

For the purpose of this study, 50 superior branches (firms) of 8 banks located in Tehran, Esfahan, Shiraz, Mashad, Tabriz, and Kerman are considered. For evaluation of these firms, the following variables play pivotal role:

$x1$: This input variable reflects the total opening hours within past 6 months;

$x2$: This input variable reflects the number of permanent staffs;

$x3$: This input variable reflects the growth in amount of long-term deposition within past 6 months;

$x4$: This input variable reflects the growth in operational cost within past 6 months;

$y1$: This output variable reflects the arithmetic average number of debit cards issued within past 6 months;

$y2$: This output variable reflects the arithmetic average number of effective transactions within past 6 months;

$y3$: This output variable reflects the growth in amount of lending within past 6 months;

$y4$: This output variable reflects the growth in amount of guarantees issued within past 6 months.

Data matrix

Firms are selected among superior branches of banks with highest number of visits since February 2012 to July 2012. These banks are commercial banks with private (3 banks), semipublic (3 banks), and state (2 banks) ownership. The information are ex- tracted from the branches' monthly reports to their local administration offices. Due to privacy policy, the authors have excused or not disseminating the detailed information. However, to provide the reader with a general overview, the statistical indices of normalized data are given in Table 1.

Table 1: Statistical indices for normalized data

	Inputs				*Outputs*			
Variables	*x*1	*x*2	*x*3	*x*4	*y*1	*y*2	*y*3	*y*4
Median	0.6000	0.4431	0.5086	0.1762	0.0675	0.0741	0.0027	0.5367
Mean	0.6189	0.4885	0.5204	0.2142	0.1674	0.1769	0.0364	0.5513
Standard Deviation	0.2245	0.2242	0.2072	0.1647	0.2454	0.2424	0.1473	0.2278
Confidence Interval	0.0158	0.0220	0.0188	0.0289	0.0479	0.0463	0.0436	0.0192

Observations:
The confidence interval were calculated with respect to Normal distribution;
The data were normalized by dividing by the largest value in each column.

Returns-to-scale: Firms vs. Industry

Table 2 shows the RTS and the efficiency score of all firms and their correspondence in *KT*. Without loss of generality, it is assumed that all firm technologies satisfy T_{VRS} . Thus, According to Li and Ng[8], the industry technology would meet the requirements for $50T_{VRS}$. As we produced some theorems, the efficiency score of each firm which are obtained by the input oriented model of BCC is identical with the efficiency score of its correspondence in *KT*, for example the efficiency score of firms 12, 20, 40 are 0.68, 1, 0.14, respectively as same as their correspondences in *KT* . The projection of DMU^{I} is K times of the projection of DMU_j .

Table 2: returns to scale and Effciency score of firms and their correspondences in KT

Firm	Returns-to-scale	Efficiency Score	Firm	Returns-to-scale	Efficiency Score
1	IRS	0.08999	26	IRS	0.15565
2	IRS	0.10512	27	IRS	0.22544
3	IRS	0.11747	28	IRS	0.20326
4	DRS	1.00000	29	IRS	0.33559
5	DRS	1.00000	30	CRS	1.00000
6	IRS	0.09039	31	IRS	0.72998
7	IRS	0.14796	32	IRS	0.16916
8	IRS	0.30246	33	IRS	0.14201
9	IRS	0.15406	34	IRS	1.00000
10	IRS	0.21626	35	DRS	1.00000
11	IRS	0.14523	36	IRS	0.24982
12	IRS	0.68401	37	IRS	0.24444
13	IRS	0.54754	38	IRS	0.17454
14	IRS	0.27531	39	IRS	0.19775
15	DRS	1.00000	40	IRS	0.14133
16	IRS	0.24128	41	IRS	0.18667
17	IRS	0.21545	42	IRS	0.54929
18	IRS	0.41932	43	IRS	0.27078
19	IRS	0.11199	44	IRS	0.17642
20	CRS	1.00000	45	DRS	1.00000
21	IRS	0.22453	46	IRS	0.14765
22	IRS	0.35759	47	IRS	0.37438
23	DRS	0.54335	48	IRS	0.18667
24	IRS	0.32693	49	IRS	0.21816
25	DRS	1.00000	50	DRS	0.54865
indusrty	IRS	0.17805			

Also the RTS status of a unit in *KT* is the same as its corresponding firm, like RTS of DMU_{35} and DMU_{35}^{I} which are DRS. In this study, the aggregated industry unit exhibits increasing returns to scale (IRS). The reference set for this industry unit is constitute by firms 20 and 34 that exhibits constant returns to scale (CRS) and IRS, respectively, so we can use its reference sets to determine the RTS of the corresponding industry unit.

Tables 3 and 4 show the slacks of firms 8, 16, 36, 46 and their correspondences in *KT* , respectively which are obtained by running model 8. As we have shown in theorem (7) the slacks of units in KT are K times of the slacks of their corresponding firms.

Table 3: Slacks of four firms

	Inputs				*Outputs*			
Firm	S_1^{-*}	S_2^{-*}	S_3^{-*}	S_4^{-*}	S_1^{+*}	S_2^{+*}	S_3^{+*}	S_4^{+*}
8	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0
36	2.33E-02	5.93E-02	0	4.42E-03	0.7681	0.7255	0.2465	0.676
46	1.87E-02	2.42E-02	0	1.23E-02	0.7184	0.721	0.2478	0.3413

Table 4: Slacks of four units in *KT*

	Inputs				*Outputs*			
Firm	S_1^{-*}	S_2^{-*}	S_3^{-*}	S_4^{-*}	S_1^{+*}	S_2^{+*}	S_3^{+*}	S_4^{+*}
8	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0
36	1.16E+00	2.96E+00	0	2.21E-01	38.405	36.275	12.325	33.8
46	9.37E-01	1.2103	0	6.13E-01	35.92	36.05	12.39	17.065

Tables 5 and 6 show the projection of 4 firms and their correspondences in *KT* , respectively. It is shown that the projection of a firm in *KT* is *K* times of the projection of its corresponding firm as we proved in theorem(7).

Table 5: Projections of four firms

	Inputs				*Outputs*			
Firm	x_1^*	x_2^*	x_3^*	x_4^*	y_1^*	y_2^*	y_3^*	y_4^*
8	0.20165	0.22276	0.17340	0.03333	0.0514	0.005	0.3229	0.20165
16	0.14477	0.11702	0.12792	0.03525	0.0532	0.0068	0.432	0.14477
36	6.00E-02	2.00E-02	0.07	4.00E-02	0.77	0.25	0.9	6.00E-02
46	6.00E-02	2.00E-02	0.07	4.00E-02	0.77	0.25	0.9	6.00E-02

Table 6: Projections of four units in *KT*

	Inputs				*Outputs*			
Firm	x_1^*	x_2^*	x_3^*	x_4^*	y_1^*	y_2^*	y_3^*	y_4^*
8	10.0825641	11.13815579	8.670067497	1.666564518	1.265	2.57	0.25	16.145
16	7.238409	5.851047275	6.396340753	1.762552592	1.975	2.66	0.34	21.6
36	3.00	1.00	3.5	2.00	40	38.5	12.5	45
46	3.00	1.00	3.5	2.00	40	38.5	12.5	45

CONCLUSION

Evaluating the performance of an aggregated industry unit is a subject which is presented by some researchers, but as we know there is no/a few article with basis of analyzing the returns to scale of an industry unit. Determining the returns to scale of an industry unit and the relation of return to scale of firms and aggregated industry unit in T_{VRS} technology needs to know the relevance of firms and their correspondences which construct firm technologies and industry technology, respectively. The current paper demonstrates the equality of the efficiency score of a special firm and its unit *KT*. Next, for an inefficient firm, we show the projection of firms in *KT* is equal to *K* times of projection of firm. Then, we prove the returns to scale status of a firm remain unchanged if we switch from *K* to *KT*. Finally, with using mentioned relations and the reference set of an aggregated unit, we specify the returns to scale of this firm and show how it can be related with firms in *KT*. To show the applicability of the presented relations and to obtain the returns to scale of an industry using the returns to scale of firms a numerical example is presented in the paper.

REFERENCE

Aparicio, J., Pastor, J. T., & Ray, S. C. (2013). An overall measure of technical inefficiency at the firm and at the industry level: the ‘lost profit on outlay’. *European Journal of Operational Research, 226*(1), 154-162.

Banker, R. D. (1981). Studies in cost allocation and efficiency evaluation.

Banker, R. D. (1984). Estimating most productive scale size using data envelopment analysis. *European Journal of Operational Research, 17*(1), 35-44.

Banker, R. D., & Thrall, R. M. (1992). Estimation of returns to scale using data envelopment analysis. *European Journal of Operational Research, 62*(1), 74-84.

Banker, R. D., Chang, H., & Cooper, W. W. (1996). Equivalence and implementation of alternative methods for determining returns to scale in data envelopment analysis. *European Journal of Operational Research, 89*(3), 473-481.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management science, 30*(9), 1078-1092.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research, 2*(6), 429-444.

Cooper, W. W., Seiford, L. M., & Tone, K. (2006). *Introduction to data envelopment analysis and its uses: with DEA-solver software and references*. Springer Science & Business Media.

Cooper, W. W., Thompson, R. G., & Thrall, R. M. (1996). Chapter 1 Introduction: Extensions and new developments in DEA. *Annals of operations Research, 66*(1), 1-45.

Färe, R., & Zelenyuk, V. (2005). On Farrell's Decomposition and aggregation. I*nternational Journal of Business and Economics, 4*(2), 167.

Färe, R., Grosskopf, S., & Li, S. K. (1992). Linear programming models for firm and industry performance. T*he Scandinavian Journal of Economics, 94*(4), 599-608.

Färe, R., Grosskopf, S., & Lovell, C. K. (1985). T*he measurement of efficiency of production* (Vol. 6). Springer Science & Business Media.

Farrell, M. J. (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society. Series A (General), 120*(3), 253-290.

Førsund, F. R., & Hjalmarsson, L. (1979). Generalised Farrell measures of efficiency: an application to milk processing in Swedish dairy plants. *The Economic Journal, 89*(354), 294-315.

Johansen, L., & Johansen, L. (1972). *Production functions; an integration of micro and macro, short run and long run aspects* (No. 04; HB141, J64.).

Leleu, H., & Briec, W. (2009). A DEA estimation of a lower bound for firms' allocative efficiency without information on price data. I*nternational Journal of Production Economics, 121*(1), 203-211.

Li, S. K., & Ng, Y. C. (1995). Measuring the productive efficiency of a group of firms. *International Advances in Economic Research, 1*(4), 377-390.

Tone, K. (1996). A simple characterization of returns to scale in DEA. J*ournal of the Operations Research Society of Japan, 39*(4), 604-613.

Tone, K., & Sahoo, B. K. (2006). Re-examining scale elasticity in DEA. *Annals of Operations Research, 145*(1), 69-87.

Zarepisheh, M., Khorram, E., & Jahanshahloo, G. R. (2010). Returns to scale in multiplicative models in data envelopment analysis. *Annals of Operations Research, 173*(1), 195-206.