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Application of the Lie Symmetry Analysis for Scond-order Fractional Differential Equations

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Abstract

Obtaining analytical or numerical solution of fractional differential equations is one of the troublesome and challenging issue among mathematicians and engineers, specifically in recent years. The purpose of this paper Lie Symmetry method is developed to solve second-order fractional differential equations, based on conformable fractional derivative. Some numerical examples are presented to illustrate the proposed approach.

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INTRODUCTION

Although solving fractional differential equations is very important, there are many fractional differential equations which can't be solved analytically. Due to this fact, finding an approximate solution of fractional differential equations is clearly an important task. In recent years, many effective methods have been proposed for finding approximate solution to fractional differential equations (Ouhadan & Elkinani, 2014; Elsaid et al., 2016; Zhanglie, 2015; Yang et al., 2014; Kumar et al., 2014; Khalil & Rashidi, 2015; Singh et al., 2016; Gaur & Singh, 2016; Gaur & Singh, 2016). The purpose of this paper Lie Symmetry method is expanding to solve fractional differential equations, based on conformable fractional derivative.

The organization of this paper is as follows: In Section 2, Conformable fractional derivative, will be described. In Section 3, Lie symmetry method for second-order fractional equations, will be explained. In Section 4, devoted to solving three second-order nonlinear fractional differential equations. Finally, discussion will be given, in section 5.

CONFORMABLE FRACTIONAL DERIVATION

Recently, conformable fractional derivative is proposed which removed some of drawbacks the presented definitions (Khalil et al., 2014; Abdeljawad, 2015)

Consider a function $f:[0,\infty)\to\mathbb{R}$. Then conformable fractional derivative of f of order α is defined by $\mathbb{R}$

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0, \alpha\in(0,1]$. If f is α- differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} T_\alpha(f)(t)$ exists, then one can define $T_\alpha(f)(0)=\lim_{t\to 0^+} T_\alpha(f)(t)$.

If the conformable derivative of f ,of order α, exists then we simply say that f is α- differentiable.

One can easily show that T_α satisfies all the following properties:

Let $\alpha \in (0,1]$ and be α-differentiable at a point $t > 0$, Then

1. For a, b$\in\mathbb{R}$ $T_\alpha(af+bg)=a\, T_\alpha(f)+b\, T_\alpha(g)$,
2. For all p$\in\mathbb{R}$ $T_\alpha(t^p)=pt^{p-\alpha}$,
3. For all constant functions $f(t)=\lambda$, $T_\alpha(\lambda)=0$,

$T_\alpha(f.g)=g.T_\alpha(f)+f.T_\alpha(g)$,

$T_\alpha(f/g)=g.T_\alpha(f)-f.T_\alpha(g))/g^2$,

$T_\alpha(f)=t^{1-\alpha}\, df/dt$.

LIE SYMMETRY METHOD FOR SECOND-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

The second-order fractional differential equations can be as following

$$T_\alpha T_\alpha y=G(t,y,T_\alpha y), \tag{1}$$

where G is a functional operator and y is an unknown function α-differentable.

Changing the independent variable as follows $x=1/\alpha\, t^\alpha$, and substitution of into Eq. 1, leads to

$$y''=F(x,y,y'), \tag{2}$$

Eq. 2 is a second-order ordinary differential equation.

Consider Eq. 2 is invariant under Lie group

$$\bar{x}=x+X(x,y)\varepsilon+O(\varepsilon^2), \quad \bar{y}=y+Y(x,y)\varepsilon+O(\varepsilon^2), \tag{3}$$

namely if be confirmed Eq. 2, then

$$\bar{y}''=F(\bar{x},\bar{y},\bar{y}'). \tag{4}$$

Substitution of the infinitesimal transformation 3 and their second-order derivative into Eq. 4 results in

$(d^2y)/(dx^2)+(\partial^2 Y/\partial x^2+[2\partial^2 Y)/\partial x\partial y-\partial^2 X/\partial x^2]\, dy/dx+[(\partial^2 Y)/(\partial y^2)-2(\partial^2 X)/\partial x\partial y](dy/dx)^2-\partial^2 X/\partial y^2\, (dy/dx)^3-[\partial Y/\partial y-2\,\partial X/\partial x]d^2y)/dx^2-3\,\partial X/\partial y\, dy/dx\, d^2y/dx^2)\varepsilon+O(\varepsilon^2)=F(x+X(x,y)\varepsilon+O(\varepsilon^2),y+Y(x,y)\varepsilon+O(\varepsilon^2),dy/dx+(\partial Y/\partial x+[\partial Y/\partial y-\partial X/\partial x]dy/dx-\partial X/\partial y(dy/dx)^2)\varepsilon+O(\varepsilon^2))$.

Expanding to order $O(\varepsilon^2)$ gives

$d^2y/dx^2+(\partial^2 Y/\partial x^2+[2\partial^2 Y/\partial x\partial y-\partial^2 X/\partial x^2]\, dy/dx+[\partial^2 Y/\partial y^2-2\ \partial^2 X/\partial x\partial y](dy/dx)^2-\partial^2\ X/\partial y^2\, (dy/dx)^3-[\partial Y/\partial y-2\ \partial X/\partial x]\ (d^2y)/(dx^2)-3\ \partial X/\partial y\, dy/dx\ d^2y/dx^2)\varepsilon+O(\varepsilon^2)=F(x,y,dy/dx)+(X\partial F/\partial x+Y\partial F/\partial y+(\partial Y/\partial x+(\partial Y/\partial y-\partial X/\partial x)dy/dx-\partial X/\partial y\, (dy/dx)^2)\ \partial F/\partial y')^{\varepsilon}+O(\varepsilon^2)$.

Discussed Lie group would be valued , if by using Eq. 2 , the following results be satisfied to $O(\varepsilon^2)$

$(\partial^2 Y)/(\partial x^2)+[2\partial^2 Y/\partial x\partial y-\partial^2 X/\partial x^2]dy/dx+[\partial^2 Y/\partial y^2-2\,\partial^2 X/\partial x\partial y](dy/dx)^2-\partial^2 X/\partial y^2\,(dy/dx)^3-[\partial Y/\partial y-2\,\partial X/\partial x]\,(d^2y)/(dx^2)-3\,\partial X/\partial y\,dy/dx\,(d^2y)/(dx^2) =$ (5)

$X\,\partial F/\partial x+Y\,\partial F/\partial y+(\partial Y/\partial x+(\partial Y/\partial y-\partial X/\partial x)\,dy/dx-\partial X/\partial y\,(dy/dx)^2)\,\partial F/(\partial y')$.

This is known as Lie's Invariance Condition, and for a given $F(x,y)$, any functions $X(x,y)$ and $Y(x,y)$ that solve Eq. 6 are the infinitesimals. Thus, if we have the infinitesimals X and Y then solving equations

$$X(x,y)\,\partial r/\partial x+Y(x,y)\,\partial r/\partial y=0,\quad X(x,y)\partial s/\partial x+Y(x,y)\,\partial s/\partial y=1, \tag{6}$$

would lead to the production alteration, that Eq. 2 converts of a second-order equation independent of s (Arrigo, 2015; Hydon, 2000; Olver, 2000).

EXAMPLES

In this section, to illustrate the proposed approach, three examples will be presented.

Example 1. Consider nonlinear fractional differential equation the following

$$T_\alpha T_\alpha y+3y\,T_\alpha y+y^3=0 \tag{7}$$

Changing the independent variable as follows $x=1/\alpha\, t^\alpha$, and substitution of into Eq. 7 results in

$$y''+3yy'+y^3=0, \tag{8}$$

where

$$F(x,y,y')=-3yy'-y^3. \tag{9}$$

By substitution of 9 into Eq. 5 , we drive

$\partial^2 Y/\partial x^2+[2\partial^2 Y/\partial x\partial y-\partial^2 X/\partial x^2]y'+[\partial^2 Y/\partial y^2-2\,\partial^2 X/\partial x\partial y](y')^2-\partial^2 X/\partial y^2 (y')^3+[\partial Y/\partial y-2\partial X/\partial x](3yy'+y^3)+3\,\partial X/\partial y\,y'\,(3yy'+y^3)+Y(3y'+3y2)+(\partial Y/\partial x+(\partial Y/\partial y-\partial X/\partial x)\,y'-\partial X/\partial y\,(y')2)3y=0$.

Setting the coefficients of y',y'^2, and y'^3 to zero, gives

$(\partial^2 Y)/(\partial x^2)+2y^3\partial X/\partial x+3y^2Y+3y\partial Y/\partial x-y^3\ \ \partial Y/\partial y=0,$

$2(\partial^2 Y)/\partial x\partial y-(\partial^2 X)/(\partial x^2)+3y\partial X/\partial x+3y^3\,\partial X/\partial y+3Y=0,$

(10)

$(\partial^2 Y)/(\partial y^2)-2\,(\partial^2 X)/\partial x\partial y+6y\,\partial X/\partial y=0,$

$(\partial^2 X)/(\partial y^2)=0$.

The solution of the system of Eq.10 leads to the infinitesimals X and Y as the following

$X=(c_1+c_2x+c_3x^2+c_4x^3+c_5x^4)y+c_6+c_7x+c_8x^2-2c_5x^3$

$Y=-(c_1+c_2x+c_3x^2+c_4x^3+c_5x^4)y^3+(c_2+2c_3x+3c_4x^2+4c_5x^3)\,y^2$

(11)

$-(2c_3+c_7+(6c_4+2c_8)x+6c_5x^2)y+4c_4+2c_8+4c_5\,x,$

where $c_1,c_2,\dots c_8$, are arbitrary constants.

For example if $c_1=1, c_2=c_3=\dots=c_8=0$, then according to Eqs. 11, the infinitesimals X and Y as follows

$$X=y,\quad Y=-y^3.$$

Substitution of this infinitesimals of Eq. 6 , and its solving leads to

$$r=x-1/y,\ s=1/2y^2, \tag{12}$$

that under this change of variables, Eq. 8 becomes

$$s''=1. \tag{13}$$

Example 2. Consider fractional differential equation the following

$$\alpha y^2 T_\alpha T_\alpha y+2t^\alpha (T_\alpha y)^3=0 \tag{14}$$

Changing the independent variable as follows $x=1/\alpha\, t^\alpha$, and substitution of into Eq. 14 yields in

$$y^2\,y''+2x(y')^3=0, \tag{15}$$

that

$$F(x,y,y')=-2x(y')^3/y^2 \tag{16}$$

By substitution of 16 into Eq. 5 , we drive

$\partial^2 Y/\partial x^2+[2\partial^2 Y/\partial x\partial y-\partial^2 X/\partial x2]y'+[\partial^2 Y/\partial y^2-2\,\partial^2 X/\partial x\partial y](y')^2-\partial^2 X/\partial y^2(y')^3+[\partial Y/\partial y-2\,\partial X/\partial x](2x(y')^3)/y^2)+3\,\partial X/\partial y\,y'(2x(y')^3)/y^2)+X\,2(y')^3)/y^2-Y(2x(y')^3)/y^3)+(\partial Y/\partial x+(\partial Y/\partial y-\partial X/\partial x)\,y'-$

$\partial X/\partial y\ (y')^2)(6x(y')^2)/y^2)=0$

Setting the coefficients of y',y'^2, and y'^3 to zero, gives

$$(\partial^2 Y)/(\partial x^2)=0,$$
$$2\ \partial^2 Y)/\partial x\partial y-(\partial(X)/(\partial x^2)=0, \qquad (17)$$

$$(\partial^2 Y)/(\partial y^2)-2\ \partial^2 X/\partial x\partial y+6x/y^2\ \partial Y/\partial x=0,$$
$$-\partial^2 Y/\partial y^2+8x/y^2\ \partial Y/\partial y-10x/y^2\ \partial X/\partial x+2/y^2 X-4x/y^3\ Y=0\ .$$

The solution of the system of Eq. 17 leads to the infinitesimals X and Y as follows

$$X=(2c_1 y+c^2/y^2)x^2+(2c_3 y^3+c_4/y^3+c_5)x+c_6\ y^2+c_7/y$$
$$Y=(c_1 y^2-c_2/y)x+c_3\ y^4+c_8\ y-c_4/y^2\quad . \qquad (18)$$

where $c_1,c_2,\ldots c_8$, are arbitrary constants.

For example if $c_5=1$, and other $c_i=0$, according to Eqs. 18 and 6 , we gives

$$r=y,\ \ s=lnx, \qquad (19)$$

that under this change of variables, Eq. 15 becomes

$$r^2\ s''+r^2\ (s')^2-2=0.$$

Example 3. Consider the following nonlinear fractional differential equation

$$\alpha T_\alpha T_\alpha\ y+yT_\alpha\ y+t_\alpha\ y4=0. \qquad (20)$$

Changing the independent variable as follows $x=1/\alpha\ t^\alpha$, and substitution of into Eq. 20 reads in

$$y''+yy'+xy4=0, \qquad (21)$$

where

$$F(x,y,y')=-(yy'+xy4\ ). \qquad (22)$$

By substitution of 21 into Eq. 5, we drive

$$\partial^2 Y/\partial x^2+[2\ \partial^2 Y/\partial x\partial y-\partial^2 X/\partial x^2]y'+[\ \ \partial^2 Y/\partial y^2-2\ \partial^2 X/\partial x\partial y](y')^2-\partial^2 X/\partial y^2(y')3-[\partial Y/\partial y-2\ \partial X/\partial x](yy'+xy4)-3\ \ \partial X/\partial y\ \ y'(yy'+xy^4)+Xy^4\ \ +Y(y'+4xy^3)+(\partial Y/\partial x+(\partial Y/\partial y-\partial X/\partial x)\ y'-\partial X/\partial y(y')\ ^2)\ y=0.$$

Setting the coefficients of y', y'^2, and y'^3 to zero, gives

$$\partial^2 Y/\partial x^2-xy^4\ (\partial Y/\partial y-2\ \partial X/\partial x)+y\ \partial Y/\partial x+Xy^4+4x\ y^3\ Y=0,$$
$$2(\partial^2 Y)/\partial x\partial y-\partial^2 X/\partial x^2+y\ \partial X/\partial x+3xy^4\ \partial X/\partial y+Y=\ 0, \qquad (23)$$

$$\partial^2 Y/\partial y^2-2\ \partial^2 X/\partial x\partial y+2y\ \partial X/\partial y=0,$$
$$\partial^2 X/\partial y^2=0\ .$$

The solution of the system of Eq. 17 leads to the infinitesimals X and Y as follows

$$X=cx,\qquad Y=-cy\ . \qquad (24)$$

By setting $c=1$, in 24 and substituting this of Eq. 6 , and its solving leads to

$$r=xy,\quad s=lnx.$$

In terms of these new variables, Eq. 21 becomes

$$s''=(r^4-r^2+2r)(s')^3+(r-3)(s')^2,$$

that is a second-order differential equation independent of s.

CONCLUSION

In this paper, Lie Symmetry Analysis method have been applied for solving fractional differential equations, based on conformable fractional derivative. Second-order fractional differential equations, have been explained by the presented method. Some examples are given for more explanation and clarification. The results showed that the presented method is easily applicable for this kind of equations.

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