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A method to obtain the best uniform polynomial approximation for the family of rational function $\frac{1}{ax^2+bx+c}$

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Abstract

In this article, by using Chebyshev's polynomials and Chebyshev's expansion, we obtain the best uniform polynomial approximation out of P_{2n} to a class of rational functions of theform $(ax^2+c)^{-1}$ on any non symmetric interval $[d,e]$. Using the obtained approximation, we provide the best uniform polynomial approximation to a class of rational functions of the form $(ax^2+bx+c)^{-1}$ for both cases $b^2-4ac\langle 0$ and $b^2-4ac\rangle 0$.

Key words: Chebyshev's polynomials, Chebyshev's expansion, uniform norm, the best uniform polynomial approximation, alternating set.

1. Introduction

$(b)\quad \sum_{j=0}^{n-1} t^{pj}T_{pj}(x)=\frac{1-t^p\cos(p\theta)-t^{pn}\cos(pn\theta)+t^{pn+n}\cos(pn\theta)\cos(p\theta)+t^{pn+n}\sin(pn\theta)\sin(p\theta)}{1+t^{2p}-2t^p\cos(p\theta)}$. In section 2, we characterize the best On of the important and applicable subjects in applied mathematics is the best approximation for functions. A large number of paper and books have considered this problem in various points of view.

Definition 1. [19] Suppose P_n denotes the space of polynomials of degree at most n, then for given $f \in C[d,e]$, there exists a unique polynomial $p_n^* \in P_n$ such that:

$$\| f - p_n^* \|_\infty \le \| f - p \|_\infty, \quad \forall p \in P_n.$$

We call p_n^* the best polynomial approximation out of P_n to f on $[d,e]$.

In other words, $p_n^* \in P_n$ is the best uniform approximation for function f on $[d,e]$ if $E_n(f,[d,e]) = \max_{d \le x \le e} | f(x) - p_n^*(x) | \le \max_{d \le x \le e} | f(x) - p(x) |, \quad \forall p \in P_n.$

The main questions of this problem are existence, uniqueness and characterization of the solution. The existence and uniqueness of the solution for the best approximation problem have been proved in [15,19].

In resent years, some researches investigated in order to characterize the best uniform approximation for special classes of functions. Several of these researches were focused on classes of functions possessing a certain expansion by Chebyshev's polynomials. For example Jokar and Mehri in [8] studied $(x-a)^{-1} (a \rangle 1)$ and $(x+1)^{-1}$. Also Achieser in [1,2] studied $(x-a)^{-1}$. Lorentz in [10] obtained the best uniform approximation for complex function $(z-\alpha)^{-1}, (z, \alpha \in C)$. In the sequel, Lubinesky in [11] showed that Lagrange interpolants at the Chebyshev zeros yield the best relevant polynomial approximation of $\left(1+(ax)^2\right)^{-1}$ on $[-1,1]$. Eslahchi and Dehghan in [6] characterized the best uniform polynomials approximation to a class of functions $\left(a^2 \pm x^2\right)^{-1}$ on $[-1,1]$ and $[-c,c]$. They also in [5] obtained the best uniform approximation to a class of $\left(T_q(a) \pm T_q(x)\right)^{-1}$. Also Elliott in [9] used the generalized form of Chebyshev's polynomials in a specific series to obtain the best approximation.

At first some definitions and theorems that will be used throughout this article are introduced.

Theorem 1. (Chebyshev's alternation theorem)[15]

Let f be in $C[d,e]$. Let the polynomial p be in P_n and $e(x) = f(x) - p_n(x)$. Then p is the best uniform approximation p_n^* to f on $[d,e]$ if and only if there exist at least $n+2$ points $x_1 \langle x_2 \langle \cdots \langle x_{n+2}$ in $[d,e]$, for which:[14]

$$|e(x_i)| = \max_{d \le x \le e} | f(x_i) - p_n(x_i) |, \text{ with } e(x_{i+1}) = -e(x_i).$$

Definition 2. [4,16] The Chebyshev's polynomial in $[-1,1]$ of degree n is denoted by T_n and is defined by $T_n(x)=\cos(n\theta)$ where $\cos\theta = x$. (1)

Note that T_n is a polynomial of degree n with leading coefficient 2^{n-1}.

Definition 3. [12] The Chebyshev's polynomial in $[d,e]$ of degree n is denoted by T_n^* and is defined by $T_n^*(x)=\cos(n\theta)$ where

$$\cos\theta = \frac{2x-(d+e)}{e-d}. \quad (2)$$

Lemma 1. [8] For $x=\cos\theta$, $|t|\langle 1$ and natural number p we have:

$$(a)\quad \sum_{j=0}^{\infty} t^{pj} T_{pj}(x) = \frac{1-t^p\cos(p\theta)}{1+t^{2p}-2t^p\cos(p\theta)},$$

uniform approximation to the class of $(ax^2+c)^{-1}$ on $[d,e]$ and in section 3, using the results from section 2, we obtain the best uniform approximation for the class of $(ax^2+bx+c)^{-1}$ on $[-1,1]$.

2. Best Approximation of $(ax^2+c)^{-1}$ on $[d,e]$

In this section, we determine the best uniform polynomial approximation out of P_{2n} to $(ax^2+c)^{-1}$ on $[d,e]$, when $\frac{c}{a}\rangle 0$ or $\frac{c}{a}\langle 0$.

Now, we prove the following lemmas to verify Chebyshev's expansion in two mentioned cases.

Lemma 2. Suppose that $x\in[d,e]$, $\frac{c}{a}\langle 0$ and $\frac{-4c}{a}\rangle (e-d)^2$. Then, we have:

$$\frac{1}{ax^2+c} = \frac{-1}{a\left(\frac{-c}{a}-x^2\right)} = \frac{-16t^2}{a(e-d)^2(t^4-1)} + \frac{32t^2}{a(e-d)^2(t^4-1)}\sum_{k=0}^{\infty} t^{2k}\overline{T}_{2k}(x); \quad (3)$$

where $t=\frac{1}{(e-d)}\left(2\sqrt{\frac{-c}{a}}-\sqrt{\frac{-4c}{a}-(e-d)^2}\right), \quad (|t|<1)$

(4)

and $\bar{T}_n(x)=\cos(n\theta)$ where $\cos\theta=\frac{2x}{e-d}$.

Proof: In the expansion of the function $\frac{1}{\alpha^2-x^2}, (\alpha^2>1)$ on $[-1,1]$ we have [17]:

$$\frac{1}{\alpha^2-x^2}=\frac{4t^2}{t^4-1}-\frac{8t^2}{t^4-1}\sum_{k=0}^{\infty}t^{2k}T_{2k}(x),$$

(5)

where, $x\in\cos\theta$ and $\alpha=\frac{t^2+1}{2t}$ and $t=\alpha-\sqrt{\alpha^2-1}$. Suppose that $\alpha=\sqrt{\frac{-c}{a}}$, then we have

$$\frac{1}{\left(\frac{t^2+1}{2t}\right)^2-\cos^2\theta}=\frac{4t^2}{t^4-1}-\frac{8t^2}{t^4-1}\sum_{k=0}^{\infty}t^{2k}T^*_{2k}(x).$$

(6)

According to (2) we can write:

$$\frac{1}{\frac{-c}{a}-x^2}=\frac{1}{\left(\frac{t^2+1}{2t}\right)^2-\cos^2\theta}=\frac{4(e-d)^2}{4(e-d)^2\left(\frac{t^2+1}{2t}\right)^2-\left(x-\frac{d+e}{2}\right)^2}.$$

(7)

Combining (6) and (7) we obtain:

$$\frac{4}{(e-d)^2\left(\frac{t^2+1}{2t}\right)^2-\left(x-\frac{d+e}{2}\right)^2}=\frac{16t^2}{(e-d)^2(t^4-1)}-\frac{32t^2}{(e-d)^2(t^4-1)}\sum_{k=0}^{\infty}t^{2k}T^*_{2k}(x).$$

(8)

Suppose that $t=\frac{1}{(e-d)}\left(2\sqrt{\frac{-c}{a}}-\sqrt{\frac{-4c}{a}-(e-d)^2}\right)$, consequently $|t|\langle 1$.(Note that for $t=\frac{2a+\sqrt{4a^2-(e-d)^2}}{(e-d)}$, the condition $|t|\langle 1$ is not true.)

Thus we have:

$$\frac{4}{\frac{-c}{a}-\left(x-\frac{d+e}{2}\right)^2}=\frac{16t^2}{(e-d)^2(t^4-1)}-\frac{32t^2}{(e-d)^2(t^4-1)}\sum_{k=0}^{\infty}t^{2k}T_{2k}^*(x).$$

(9)

where with $\overline{T}_n(x)=\cos(n\theta)$, $\cos\theta=\frac{2x}{e-d}$, so relation (3) is proved. □

Lemma 3. Suppose that $x\in[d,e]$ and $\frac{c}{a}\rangle 0$. Then we have:

$$\frac{1}{ax^2+c}=\frac{1}{a\left(x^2+\frac{c}{a}\right)}=\frac{16t^2}{a(e-d)^2(t^4-1)}-\frac{32t^2}{a(e-d)^2(t^4-1)}\sum_{k=0}^{\infty}(-1)^k t^{2k}\overline{T}_{2k}(x);$$

(10)

where

$$t=\frac{1}{(e-d)}\left(\sqrt{\frac{4c}{a}+(e-d)^2}-2\sqrt{\frac{c}{a}}\right),\quad (|t|<1).$$

(11)

and $\overline{T}_n(x)=\cos(n\theta)$ where $\cos\theta=\frac{2x}{e-d}$.

Proof: In the expansion of the function $\frac{1}{\beta^2+x^2}$ on $[-1,1]$ we have[6]:

$$\frac{1}{\beta^2+x^2}=\frac{4t^2}{(t^4-1)}-\frac{8t^2}{(t^4-1)}\sum_{k=0}^{\infty}(-1)^k t^{2k}T_{2k}^*(x),$$

(12)

where $x=\cos\theta$, $\beta=\frac{1-t^2}{2t}$ .With suppose $\beta=\sqrt{c/a}$,the rest of proof is similar to the proof of lemma 2. Thus we omit it.□

Theorem 2. The best uniform polynomial approximation out of P_{2n} for $(ax^2+c)^{-1}$ where $\frac{c}{a}\langle 0$, on $[d,e]$ and $\frac{-4c}{a}\rangle(e-d)^2$, is as follows:

$$p_{2n}^*(x)=\frac{-16t^2}{a(e-d)^2(t^4-1)}+\frac{32t^2}{a(e-d)^2(t^4-1)}\sum_{k=0}^{n-1}t^{2k}\overline{T}_{2k}(x)-\frac{32t^{2n+2}}{a(e-d)^2(t^4-1)^2}\overline{T}_{2n}(x),$$
(13)

and $$E_{2n}(f)=\left\|f-p_{2n}^*\right\|_\infty=\frac{32t^{2n+4}}{|a|(e-d)^2(t^4-1)^2},$$

(14) where $t=\frac{1}{(e-d)}\left(2\sqrt{\frac{-c}{a}}-\sqrt{\frac{-4c}{a}-(e-d)^2}\right)$, $(|t|\langle 1)$, $\overline{T}_n(x)=\cos(n\theta)$

where $\cos\theta=\frac{2x}{e-d}$.

Proof: Noting to Chebyshev's alternation theorem we should prove that the function $e_{2n}(x)=\frac{1}{ax^2+c}-p_{2n}^*(x)$
(15)

has $2n+2$ alternating points in $[d,e]$. From (3) and (13) we have:

$$e_{2n}(x)=\frac{32t^2}{a(e-d)^2(t^4-1)}\sum_{k=n}^{\infty}t^{2k}\overline{T}_{2k}(x)+\frac{32t^{2n+2}}{a(e-d)^2(t^4-1)^2}\overline{T}_{2n}(x).$$
(16)

By replacing $p=2$ in lemma 1 and subtracting both sides of (b) from (a) we obtain:

$$\sum_{k=n}^{\infty}t^{2k}\overline{T}_{2k}(x)=t^{2n}\,\frac{\cos(2n\theta)-t^2(\cos(2n\theta)\cos(2\theta)+\sin(2n\theta)\sin(2\theta))}{1+t^4-2t^2\cos(2\theta)}.$$
(17)

By replacing (17) in (16), we obtain:

$$e_{2n}(x)=\frac{32t^{2n+2}}{a(e-d)^2(t^4-1)^2}\left[\left\{\frac{(1-t^2\cos(2\theta))(t^4-1)+(1+t^4-2t^2\cos(2\theta))}{1+t^4-2t^2\cos(2\theta)}\right\}\cos(2n\theta)\right.$$
$$\left.+\left\{\frac{t^2(t^4-1)\sin(2\theta)}{1+t^4-2t^2\cos(2\theta)}\right\}\sin(2n\theta)\right]$$
(18)

Noting to (4), we have $\frac{-c}{a}=\left(\frac{(e-d)(t^2+1)}{4t}\right)^2$. Then we can rewrite (18) in the form of:

$$e_{2n}(x)=\frac{32t^{2n+4}}{a(e-d)^2(t^4-1)^2}\left\{\frac{(e-d)^2\left(\frac{-c}{a}\right)-x^2\left(\frac{-8c}{a}-(e-d)^2\right)}{(e-d)^2\left(\frac{-c}{a}-x^2\right)}\cos(2n\theta)\right.$$
$$\left.+\frac{2\sqrt{\frac{4c^2}{a^2}+\frac{c}{a}(e-d)^2}\sqrt{x^2(e-d)^2-4x^4}}{(e-d)^2\left(\frac{-c}{a}-x^2\right)}\sin(2n\theta)\right\}.$$
(19)

Now if we define:

$$F_1(x)=\frac{(e-d)^2\left(\frac{-c}{a}\right)-x^2\left(\frac{-8c}{a}-(e-d)^2\right)}{(e-d)^2\left(\frac{-c}{a}-x^2\right)}, \tag{20}$$

$$F_2(x)=\frac{2\sqrt{\frac{4c^2}{a^2}+\frac{c}{a}(e-d)^2}\sqrt{x^2(e-d)^2-4x^4}}{(e-d)^2\left(\frac{-c}{a}-x^2\right)}. \tag{21}$$

Then we have: $F_1'(x)=\dfrac{-2x\left(\frac{8c^2}{a^2}+\frac{2c}{a}(e-d)^2\right)}{(e-d)^2\left(\frac{-c}{a}-x^2\right)^2}.$

(22)

It is easy to conclude that if $0 \notin [d,e]$, then $F_1(x)$ is a monotonic function and if $0 \in [d,e]$ then $F_1(x)$ is a monotonic function on each interval $[d,0]$, $[0,e]$. Also we have: $F_1^2(x) + F_2^2(x) = 1, \qquad x \in (d,e]$.
(23)

Therefore, according to (22) and definition of $\overline{T}_n$ for $x \in [d,e]$, we have:
$-1 \le F_1(x) \le 1$. Hence according to mean value theorem for every $x \in [d,e]$, there exists a $\eta \in (0,\pi)$ such that $\quad \cos\eta = F_1(x), \qquad x \in [d,e]$.
(24)

Therefore, from (23) we can write: $\sin\eta = F_2(x)$.
(25)

Replacing (24) and (25) in (19) we obtain:

$$e_{2n}(x) = \frac{32t^{2n+4}}{a(e-d)^2(t^4-1)^2}\cos(\eta + 2n\theta).$$
(26)

Now, if x varies from d to e, then $\cos(2n\theta+\eta)$ varies from $\cos(n+1)\pi$ to $\cos(-\pi)$ and consequently $\cos(2n\theta+\eta)$ possesses at least $2n+2$ extermal points, where it assumes alternately the values ± 1. Therefore p_{2n}^* is the best approximation out of P_{2n}, and (14) will be proved with considering (26) .□

Theorem 3. The best uniform polynomial approximation out of P_{2n} for $(ax^2+c)^{-1}$ where $\frac{c}{a} \rangle 0$, on $[d,e]$ will be:

$$p_{2n}^*(x) = \frac{16t^2}{a(e-d)^2(t^4-1)} - \frac{32t^2}{a(e-d)^2(t^4-1)}\sum_{k=0}^{n-1}(-1)^k t^{2k}\overline{T}_{2k}(x) - \frac{32(-1)^n t^{2n+2}}{a(e-d)^2(t^4-1)^2}\overline{T}_{2n}(x), (27)$$

and $$E_{2n}(f) = \left\| f - p_{2n}^* \right\|_\infty = \frac{32t^{2n+4}}{|a|(e-d)^2(t^4-1)^2}.$$

(28) where $t = \frac{1}{(e-d)}\left(\sqrt{\frac{4c}{a} + (e-d)^2} - 2\sqrt{\frac{c}{a}}\right), (|\,t\,|<1),\ \overline{T}_n(x) = \cos(n\theta)$

where $\cos\theta = \frac{2x}{e-d}$.

Proof: The proof is similar to the proof of theorem 2.□

If we obtain the best uniform polynomial approximation for $f(x)=\dfrac{1}{25-x^2}$ on $[-3,3]$ by using theorem 2**,** with $n=3$, $p_6^*(x)$ we will see that this result is similar to the best uniform polynomial approximation obtained in [6]. If we obtain the best uniform polynomial approximation for $f(x)=\dfrac{1}{25+x^2}$ on $[-5,5]$ by using theorem 3**,** with $n=2$, $p_4^*(x)$ we will see that this result is similar to the best uniform polynomial approximation obtained in [6].

Example 1. In figure 1, the function $f(x)=\dfrac{1}{-2x^2+19}$ has been drawn. The dashed points show the best uniform polynomial approximation of degree 8 , $p_8^*(x)$, to $(-2x^2+19)^{-1}$ on $[-1,2]$.

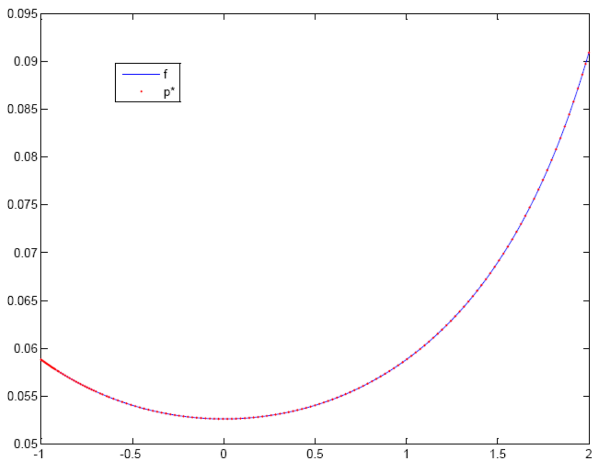

Figure 1: The best approximation of $(-2x^2+19)^{-1}$.

Example 2. In figure 2, both the function and its best uniform polynomial approximation, $p_{16}^*(x)$, (the dashed point) of degree 16 to $(5+x^2)^{-1}$ on $[0,2]$ has been shown.

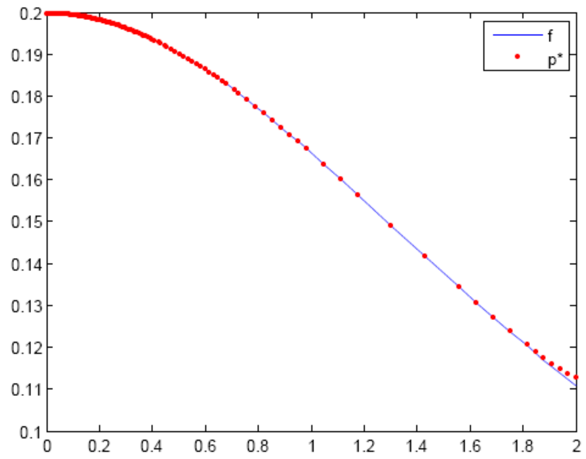

Figure 2: The best approximation of $(5+x^2)^{-1}$.

3. Best Approximation of $\left(ax^2+bx+c\right)^{-1}$

In this section, by using the previous theorems, we obtained the best polynomial approximation for $\left(ax^2+bx+c\right)^{-1}$ on $[-1,1]$.

Theorem 4. The best uniform polynomial approximation out of P_{2n} for $\left(ax^2+bx+c\right)^{-1}$ on $[-1,1]$ is as follows:

$$(a)\quad p_{2n}^{*}(x)=\frac{1}{a}\left[\frac{-4t^2}{(t^4-1)}+\frac{8t^2}{(t^4-1)}\sum_{k=0}^{n-1}t^{2k}T_{2k}\left(x+\frac{b}{2a}\right)-\frac{8t^{2n+2}}{(t^4-1)^2}T_{2n}\left(x+\frac{b}{2a}\right)\right], \tag{29}$$

where,

$$t=\sqrt{\frac{b^2-4ac}{4a^2}}-\sqrt{\frac{b^2-4ac}{4a^2}-1},\quad \left(b^2-4ac>4a^2>0\right),|t|\langle 1. \tag{30}$$

$$(b)\quad p_{2n}^{*}(x)=\frac{1}{a}\left[\frac{4t^2}{(t^4-1)}-\frac{8t^2}{(t^4-1)}\sum_{k=0}^{n-1}t^{2k}(-1)^kT_{2k}\left(x+\frac{b}{2a}\right)+\frac{8(-1)^nt^{2n+2}}{(t^4-1)^2}T_{2n}\left(x+\frac{b}{2a}\right)\right], \tag{31}$$

where,

$$t=\sqrt{\frac{-b^2+4ac}{4a^2}+1}-\sqrt{\frac{-b^2+4ac}{4a^2}},\qquad b^2-4ac\langle 0,|t|\langle 1. \tag{32}$$

Proof: We can write the function $\left(ax^2+bx+c\right)^{-1}$ in the form of:

(33)

$$\frac{1}{ax^2+bx+c}=\frac{1}{a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right)}.$$

Since $x\in[-1,1]$ therefore

$$(34)\ x+\frac{b}{2a}\in\left[-1+\frac{b}{2a},1+\frac{b}{2a}\right]=[d,e].$$

Now, by changing x to $x+\frac{b}{2a}$ in theorems 2 and 3, we have:

$$\overline{T}_{2n}\left(x+\frac{b}{2a}\right)=\cos\left(2n\arccos\frac{2\left(x+b/2a\right)}{1+b/2a+1-b/2a}\right)=\cos\left(2n\arccos\left(x+b/2a\right)\right)=T_{2n}\left(x+\frac{b}{2a}\right).$$

Case1 : ($b^2-4ac>0$) In this case, replacing $\frac{-c}{a}$ by $\frac{b^2-4ac}{4a^2}$, according to (34), the defined t in theorem 2, changes to (30) where $b^2-4ac>4a^2$. Therefore, we can prove (a) by using theorem 2.

Case2 : ($b^2-4ac\langle 0$) In this case, replacing $\frac{c}{a}$ by $\frac{-b^2+4ac}{4a^2}$, according to (34), the defined t in theorem 3, changes to (32). Therefore, we can prove (b) by using theorem 3.

Example 3. In figure 3, we have drawn the function $f(x)=\frac{1}{x^2+2x-15}$. Also, the dashed points show the best uniform polynomial approximation of degree 6 , $p_6^*(x)$, to $(x^2+2x-15)^{-1}$ on $[-1,1]$.

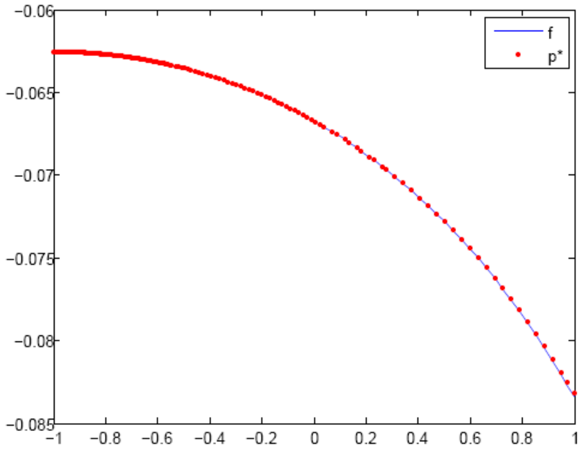

Figure 3: The best approximation of $(x^2+2x-15)^{-1}$.

Example 4. In figure 4, both the function and its best uniform polynomial approximation , $p_{16}^*(x)$, (the dashed point) of degree 16 to $(x^2-2x+6)^{-1}$ on $[-1,1]$ are shown.

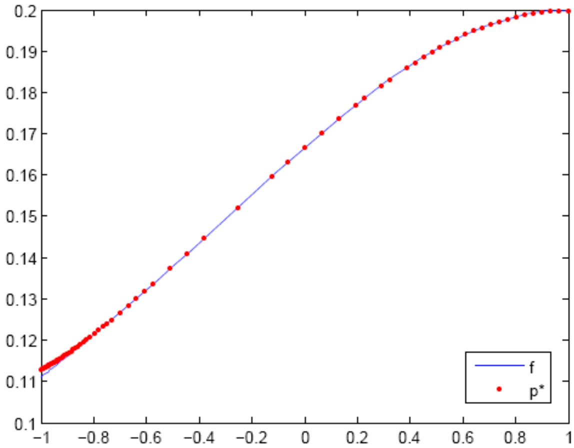

Figure 4: The best approximation of $(x^2-2x+6)^{-1}$.

4. Conclusion

As seen in this article, in the sequel of previous researches, the best uniform approximation for $(ax^2 \pm c)^{-1}$ was achieved. In this case, we applied the interval $[d,e]$ as general in place of $[-1,1]$.

Also, by characterizing the best uniform approximation for $\left(ax^2+bx+c\right)^{-1}$ on $[-1,1]$, a more general form than previous approximation in [6,8] was obtained.

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