Unweighted p-center problem on extended stars

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Abstract

An extended star is a tree which has only one vertex with degree larger than two. The \( p \)-center problem in a graph \( G \) asks to find a subset \( X \) of the vertices of \( G \) of cardinality \( p \) such that the maximum weighted distances from \( X \) to all vertices is minimized. In this paper we consider the \( p \)-center problem on the unweighted extended stars, and present some properties to find solution.

Keywords: Location theory, center problem, extended star

1. Introduction

Let \( G=(V, E) \) be an undirected graph with vertex set \( V \) and edge set \( E \). Each vertex \( v_i \) has a positive weight \( w_i \) and the edges of graph have positive lengths. An important problem in the location theory is the \( p \)-center problem. In the \( p \)-center problem we want to find a subset \( X \subseteq V \) of cardinality \( p \) such that the maximum weighted distances from \( X \) to all vertices is minimized. If all the weights are equal the problem is called unweighted \( p \)-center problem.
The \( p \)-Center problem has been known to be NP-hard, [5]. Lan et al. in [6] presented a linear-time algorithm for solving the 1-center problem on weighted cactus graphs. Frederickson in [4] solved this problem for trees in optimal linear-time (without necessarily restricting the location of the facilities to the vertices of the tree) using parametric search. Bespamyatnikh et al. in [1] gave an \( O(pn) \) time algorithm for this problem on circular-arc graphs. Kariv and Hakimi in [5] addressed the \( p \)-center problem on general graphs. In [8], Tamir showed that the weighted and unweighted \( p \)-center problems in networks can be solved in \( O(n^p m^p \log^2 n) \) time and \( O(n^{p-1} m^p \log^3 n) \) time, respectively. Burkard and Dollani [2] considered the case that some vertices have negative weights and presented an \( O(n^2 \log n) \) algorithm for \( p \)-center problem on a tree. They also presented a linear time algorithm for 1-center problem with pos/neg weights on paths and star graphs. For further literature on the \( p \)-median (and center) problem the reader is referred to the books of Mirchandani and Francis [7] and Drezner and Hamacher [3].

In what follows we state the \( p \)-center problem on a graph in Section 2. In Section 3 we show the unweighted \( p \)-center problem on a path can be solved in a constant time. Section 4 contains some properties of unweighted \( p \)-center problem on extended stars that leads to find a solution.

2 Problem formulation

Let \( G= (V,E) \) be a graph, where \( V \) is the set of vertices with \( |V|= n \), and \( E \) is the set of edges. Every edge with the end vertices \( u \) and \( v \) is presented by \( e_{uv} \). We assume the weights of all vertices and edges are the same and equal to one.

In the \( p \)-center problem the maximum of the distances is minimized over all \( X \subseteq G \) with \( |X|= p \), i.e.

\[
\min_{X \subseteq G} d(X,v_i) = \max_{i=1,\ldots,n} d(X,v_i),
\]

where \( d(X,v_i) = \min_{x \in X} d(x,v_i) \) and \( d(x,v) \) is the minimum distance between \( x \) and \( v \) in \( G \).

In this paper we consider the case that \( G \) is a extended star. An extended star is a tree which has only one vertex with degree larger than two. We call this vertex with degree larger than two as central vertex.

3 The \( p \)-center on a path
Let $P$ be a path with vertex set \{v_1, v_2, \ldots, v_n\}, where $v_i$ is adjacent to $v_{i+1}$ for $i = 1, 2, \ldots, n-1$. The following results are straightforward, and so we omit a proof.

- The solution of 1-center problem on $P$ is vertex $v_{\left\lfloor \frac{n}{2} \right\rfloor}$.
- The solution of 2-center problem on $P$ is vertices $v_{\left\lfloor \frac{n}{4} \right\rfloor}$ and $v_{\left\lfloor \frac{3n}{4} \right\rfloor}$.
- In general the solution of $p$-center problem on $P$ is vertices $v_{\left\lfloor \frac{(2i-1)n}{2p} \right\rfloor}$ for $i = 1, \ldots, p$.

Using above statements we can find a solution on a path in a constant time, i.e.:

**Theorem 3.1** The unweighted $p$-center problem on a path can be solved in $O(1)$ time.

**Example 3.2** Consider the path depicted in Figure 3.2 which all its weights are equal to one. Table 3.2 contains the solutions of unweighted $p$-center problem on this path for different values of $p$. The solutions are computed using the statement 3. For example for $p = 4$ the solution is $X^* = \{x_1, x_2, x_3, x_4\}$ where

\[
    x_1 = v_{\left\lfloor \frac{14}{8} \right\rfloor} = v_2,
\]

\[
    x_2 = v_{\left\lfloor \frac{8}{8} \right\rfloor} = v_6,
\]

\[
    x_3 = v_{\left\lfloor \frac{3\times14}{8} \right\rfloor} = v_9,
\]

and

\[
    x_4 = v_{\left\lfloor \frac{7\times14}{8} \right\rfloor} = v_{13}.
\]
4 The $p$-center on extended stars

Now consider the $p$-center problem on extended stars. We state some properties to decrease computation for finding the solution.

**Theorem 4.1** Let $S$ be an extended star and $S' \subseteq S$ be a sub extended star of $S$ contains the $p$ longest branches of $S$ then the solution of the $p$-center problem on $S'$ for $p > 1$ is also a solution of this problem on $S$.

**Proof.** Let $X = \{c_1, \ldots, c_p\}$ be a solution of $p$-center problem on $S$. If the number of branches in $S$ is less than or equal to $p$ then $S = S'$ and the theorem holds. Otherwise let $c_j$ be a center on branch $B_j$ where $B_j$ is not in the $p$ longest
branches of $S$. Also there is a branch $B_j$ which is one of the $p$ longest branches of $S$ and not contains any center $c_r$, $r = 1, \ldots, p$. Let $o$ be the unique vertex of $S$ with $\text{deg}(o) > 2$. If $o$ is assigned to $c_i$ then all vertices on $B_j$ are also assigned to $c_i$ and since $|B_j| < |B_i|$ we can decrease the value of objective function by moving $c_i$ on $B_i$ in the direction of $o$. Which contradicts that $X$ is a solution of $p$-center problem. In the other case if $o$ is assigned to $c_k \neq c_i$. Then all vertices on $B_j$ are also assigned to $c_k$, specially the end vertex $v_i$ of $B_j$. Since $d(v_i,c_k) > d(v_m,o)$ where $v_m$ is end vertex of $B_i$, we can set $o$ in $X$ instead of $c_i$ which does not cause increasing the objective function. By now we showed that there exist a solution $X = \{c_1, \ldots, c_p\}$ such that for $r = 1, \ldots, p$ $c_r$ is in the one of $p$ longest branches of $S$. Now let $S'$ be the sub extended star of $S$ contains $p$ longest branches. Assignment vertices in $S'$ is the same as $S$. Suppose $o$ is assigned to $c_h$ then any vertex $v \in S \setminus S'$ is also assigned to $c_h$ so if we delete the vertices in $S \setminus S'$ the solution does not change, just the value of objective function will be increased. This complete the proof. 

□

Using Theorem 4.1 in the cases $p = 2$ the solutions lies on the longest path or diameter of star so the problem reduces to finding solution on the longest path. Also for the case $p = 1$ the solution lies on the longest path therefore using statements 1 and 2, we can state the following theorem.

**Theorem 4.2** Let $S$ be an extended star which its diameter be the path $P = v_1, v_2, \ldots, v_d$ with length $d$. The solution of the 1-center problem is $v_{\left\lceil \frac{d+1}{2} \right\rceil}$ and the solutions of the 2-center problem are $v_{\left\lceil \frac{d+1}{4} \right\rceil}$ and $v_{\left\lceil \frac{3(d+1)}{4} \right\rceil}$.

**Theorem 4.3** There is a solution $X = \{c_1, \ldots, c_p\}$ of the $p$-center problem on extended star $S$ such that for $i = 1, \ldots, p$ $c_i$ lies on a branch of $S$ which its length greater than or equal to $\frac{d}{2p}$.

**Proof.** Let $T$ be an extended star with central vertex $O$. By Theorem 4.1, there is a solution of the $p$-center problem on a sub extended star containing the
Let $X = \{ c_1, c_2, \ldots, c_p \}$ be a solution. Suppose $L_i$ is a branch of $T$ with length less than $\frac{d}{2p}$, and let $c_i \in L_i$. Let $P$ be the longest path in $T$ of length $d = \text{diam}(T)$. We consider the following cases:

**Case 1.** $O \notin X$. Let $X_1 = (X \setminus \{ c_i \}) \cup \{ O \}$. We show that $X_1$ is a solution. Since $c_i \in L_i$, $|P \cap X| \leq p - 1$. Let $Y = X \cap P$, and let $x = \max \text{mind}(v, c_i)$, where $c_i \in Y$, and $v \in P - Y$. We observe that $(p - 1)x \geq d$, and so $x \geq \frac{d}{2p - 2}$. This means that there is a vertex $v$ on $P$ such that the minimum distance from $v$ to $Y$ is at least $\frac{d}{2p - 2}$. Since $L_i$ is a branch with length less than $\frac{d}{2p}$, replacing $c_i$ by $O$ does not reduce the maximum distance of a vertex outside $X$ to $X_1$. This means that $X_1$ is also a solution.

**Case 2.** $O \in X$. Let $w$ be a vertex in $P \setminus X$ in a minimum distance from $O$, and let $X_2 = (X \setminus \{ c_i \}) \cup \{ w \}$. Similar to case 1 we observe that $X_2$ is a solution.

We continue the above process until there is no branch $L_i$ with length less than $\frac{d}{2p}$ such that $X \cap L_i \neq \emptyset$. □

Note that by using the Theorems 4.1 and 4.3 we can eliminate some branches and solve the $p$-center on the remaining sub-tree. The solution will be the same. So the computation will be reduced.

**Example 4.4** Consider the extended star depicted in Figure 4.4 which all its weights are equal to one. Table 4.4 shows the branches that we consider to solve unweighted $p$-center problem for different values of $p$. 

Figure 2: The extended star for Example 4.4

Table 2: The considered branches to solve the unweighted $p$-center problem on extended star.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$S'$</th>
<th>Value of objective function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${ B_1 }$</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>${ B_1, B_2 }$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>${ B_1, B_2, B_3 }$</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>${ B_1, B_2, B_3, B_6 }$</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>${ B_1, B_2, B_3, B_6, B_7 }$</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>${ B_1, B_2, B_3, B_6, B_7, B_8 }$</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>${ B_1, B_2, B_3, B_6, B_7, B_8 }$</td>
<td>2</td>
</tr>
</tbody>
</table>
5 Summary and conclusion

We considered the unweighted $p$-center problem on paths and extended stars. For the case path we presented an $O(1)$ time algorithm and for extended stars some property are presented to reduce computations.

References


