On Bottleneck Product Rate Variation Problem with Batching

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Abstract

The product rate variation problem minimizes the variation in the rate at which different models of a common base product are produced on the assembly lines with the assumption of negligible switch-over cost and unit processing time for each copy of each model. The assumption of significant setup and arbitrary processing times forces the problem to be a two phase problem. The first phase determines the size and the number of batches and the second one sequences the batches of models.

In this paper, the bottleneck case i.e. the min-max case of the problem with a generalized objective function is formulated. A Pareto optimal solution is proposed and a relation between optimal sequences for the problem with different objective functions is investigated.

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1. **Introduction**

The product rate variation problem (PRVP) minimizes the variation in the rate at which different models of a common base product are produced on the assembly lines [6]. The problem minimizes both the earliness and the tardiness penalties that respond to the customer demands for a variety of models without holding large inventories or incurring large shortages. This is a problem of finding a sequence of different models distributed as evenly as possible on the assembly lines with the assumption of negligible switch-over cost and unit processing time for each copy of each model.

The problem has been formulated as a non-linear integer programming with the objective of minimizing the deviation between the actual and the ideal production under the assumption that the system has sufficient capacity with negligible changeover costs from one model to another and each model is produced in a unit time [12, 13]. The problem has mathematically interesting base model with theoretical value and real world applications, see [3].

The problem has been extensively studied and solved in pseudo-polynomial time. The total PRVP i.e. the problem with the objective of minimizing the total deviation has been solved in $O(D^3)$, [9, 10] and the problem with the objective of minimizing the maximum deviation i.e. the bottleneck PRVP in $O(DlogD)$ time.

The bottleneck PRVP with absolute-deviation objective has been solved in [15] and with square-deviation objective in [4]. Recently a solution for the bottleneck PRVP with a general objective has been proposed with a relation between optimal sequences for the problem with different objectives [5].

The assumption of negligible change-over costs from one model to another cannot be undertaken if different models have significant setup and arbitrary processing times. It has been investigated that a sequence, of different models having significantly different setup and processing times, obtained by the methods designed for synchronized assembly lines is as good as a randomly obtained sequence, see [11].

Significant setup and arbitrary processing times can be undertaken when the planning horizon is partitioned into a finite number of time-buckets with equal length. The time length of a time-bucket is called a takt-time. A time-bucket consists of a setup and a batch (a copy or several copies) of a model. The assumption that allows significant setup and arbitrary processing times forces the product rate variation problem to be a two-phase problem [16, 17]. We call this

problem as the product rate variation problem with batching. The first phase is the batching problem that determines the batch size and the number of batches of the models. The second phase is the sequencing problem that sequences the batches. The problem that determines the size and the number of batches and minimizes the maximum variation in the rate at which batches of different models of a common base product are produced is called the bottleneck product rate variation with batching problem.

In this paper, the bottleneck product rate variation problem with batching is formulated. Here, bottleneck means the minimization of the maximum deviation between the actual and the ideal production. Note that the total product rate variation problem with batching exists in the literature [11]. A Pareto optimal solution to the problem is described. The first phase problem is solved with an algorithm appeared in [11] then the second phase is solved with bottleneck assignment or perfect matching with a bisection search methods. A relation between optimal sequences is investigated.

The plan of the paper is as follows. Section 2 reviews the mathematical model. In Section 3, a Pareto optimal solution is described. An algorithm for the first phase problem and two sequencing procedures for the second phase problem are discussed. Section 4 shows the relation between optimal sequences. The last section concludes the paper.

2. Mathematical Models

Let s_i and p_i be the setup and processing times of a model i, $i = 1,...,n$, respectively. The total demands $D = \sum_{i=1}^{n} d_i$ are manufactured over the planning horizon T partitioned into $\mathcal{D}$ time- buckets i.e. the number of batches with the takt-time $t = \frac{T}{D}$. There may exist batches with no models to be manufactured. Such empty batches are potentially useful for the improvement of the system performance [11]. The takt-time satisfies $t \geq s_i + p_i, i = 1,....,n$. For feasible t, the number of batches of model i is $\partial_i = \left\lceil \dfrac{d_i}{\left\lfloor \frac{\frac{T}{i}-s_i}{p_i} \right\rfloor} \right\rceil$ such that $\left(\mathcal{D} - \sum_{i=1}^{n} \partial_i \right) = \partial_0 \geq 0$,

where ∂_0 is denoted to be the number of empty batches.

There exist unavoidable time lost in each batch due to the variability of setup and processing times. A model i, $i = 1,\ldots,n$ spends $d_i p_i$ time units out of $\partial_i t$ in all batches. The total time lost $\sum_{i=1}^{n}(\partial_i t - d_i p_i) = T\left(1 - \frac{\partial_0}{D}\right) - \sum_{i=1}^{n} d_i p_i$ is to be minimized. We minimize the total time lost maximizing $\frac{\partial_0}{D}$. The minimum average response time $\frac{D}{\partial_0}$ improves the system responsiveness and minimum work-in-process inventory is achieved maximizing D, [11].

The multi-objective mathematical programming for the batching phase of the product rate variation problem with batching [16, 11], is

$$\text{maximize} \qquad \frac{\partial_0}{D} \qquad\qquad (1)$$

$$\text{maximize} \qquad \partial_0 \qquad\qquad (2)$$

$$\text{maximize} \qquad D \qquad\qquad (3)$$

subject to

$$\partial_i = \left| \left\lceil \frac{d_i}{\frac{T-s_i}{p_i}} \right\rceil \right| \qquad\qquad i = 1,\ldots,n \qquad (4)$$

$$\left(D - \sum_{i=1}^{n} \partial_i\right) = \partial_0 \geq 0 \qquad\qquad (5)$$

$$\frac{T}{\max_i \{s_i + p_i\}} \geq D \qquad\qquad i = 1,\ldots,n \qquad (6)$$

$$D \geq 0, \qquad\qquad \text{integer} \qquad (7)$$

Constraint (4) ensures that the number of batches does not exceed the capacity of the system. Constraint (5) shows that the demand of each model is met exactly. Constraints (5), (6) and (7) ensure the feasibility of $\mathcal{D}$.

We denote $\tilde{x}_{ik}$, $i = 0, 1, \ldots, n$; $k = 1, \ldots, \mathcal{D}$ to be the actual cumulative number of batches for model i produced during the time-buckets 1 through k. The actual cumulative production of model i during the same time-buckets is $\gamma_i \tilde{x}_{ik}$, where $\gamma_i = \frac{d_i}{\tilde{d}_i}$ is the average number of copies of model i per batch. The ideal cumulative production of model i during 1 through k time-buckets is $\gamma_i k \tilde{r}_i$, where $\tilde{r}_i = \frac{\tilde{d}_i}{\mathcal{D}}$ is the batch rate. The sequencing problem minimizes the deviation between the actual and the ideal productions. Let m be a positive integer.

The mathematical programming for the sequencing phase of the bottleneck product rate variation problem with batching is

$$\text{minimize} \left[F_m = \max_{i,k} \left(\gamma_i |\tilde{x}_{ik} - k\tilde{r}_i| \right)^m \right] \qquad (8)$$

subject to

$$\sum_{i=1}^{n} \tilde{x}_{ik} = k, \qquad k = 1, \ldots, \mathcal{D} \qquad (9)$$

$$\tilde{x}_{i(k-1)} \leq \tilde{x}_{ik}, \qquad i = 0, 1, \ldots, n; \ k = 2, \ldots, \mathcal{D} \quad (10)$$

$$\tilde{x}_{i\mathcal{D}} = \tilde{d}_i, \ \tilde{x}_{i0} = 0, \ i = 0, 1, \ldots, n \qquad (11)$$

$$\tilde{x}_{ik} \geq 0, \ \text{integer} \quad i = 0, 1, \ldots, n; \ k = 1, \ldots, \mathcal{D} \qquad (12)$$

Constraint (9) ensures that exactly k batches are produced during the periods 1 through k. Constraint (10) states that the total number of batches is a nondecreasing function of k. Constraint (11) guarantees that the batches are exactly met. Constraints (9), (10) and (12) ensure that exactly one batch of a model is sequenced during a unit time-bucket. Note that the formulation of the sequencing phase of the problem is similar to the formulation in [15] for the bottleneck product rate variation problem.

We denote Problem F_m for the bottleneck product rate variation problem with batching with the objective function Fm and the constraints (9) to (12).

3. Pareto Optimal Solution

The batching phase of the problem is solved with an algorithm appeared in [11]. The algorithm finds $D_{max} = \dfrac{T}{max_j\{s_j + p_j\}}$, the largest possible number of batches, then enumerates all possible number of batches with $n \leq D \leq D_{max}$. The minimum number of batches $\check{d}_i = \left\lceil \dfrac{d_i}{\frac{T}{D} - s_i} \right\rceil$ is calculated for each D.

The algorithm determines the number and the size of batches in $O(nD)$ time. The sequencing phase of the problem is solved either using bottleneck assignment method or perfect matching with a bisection search method.

3.1. Bottleneck Assignment Method

Problem F_m can be solved transforming the problem into an equivalent bottleneck assignment problem. The bottleneck assignment problem is

$$\text{minimize } \max_{i,j,k} B_{ijk} x_{ijk} \tag{13}$$

subject to

$$\sum_{i=0}^{n} \sum_{j=1}^{d_i} x_{ijk} = 1, k = 1, \dots, \mathcal{D} \tag{14}$$

$$\sum_{k=1}^{\mathcal{D}} x_{ijk} = 1, i = 0,1, \dots, n; k = 1, \dots, \mathcal{D} \tag{15}$$

where $B_{ijk} = \max\{(\gamma_i|j-1-(k-1)\tilde{r}_i|)^m, (\gamma_i|j-k\tilde{r}_i|)^m\}$, $i = 0,1,\dots,n$; $j = ,\dots,d_i$; $k = 1,\dots,\mathcal{D}$ is the bottleneck assignment cost for (i,j) i.e. the jth batch of model i when assigned to k time- bucket and $x_{ijk} = 1$ if (i,j) is assigned to k or 0, otherwise.

The constraint (14) shows that exactly one batch is produced in one time-bucket and the constraint (15) ensures that one batch is produced exactly once. There exists an optimal solution $\{x_{ijk}\}$ that preserves order [11].

A number of algorithms solve the bottleneck assignment problem of Problem Fm [14]. The Hungarian method takes $O(\mathcal{D}^3)$ time to solve the problem with $2\mathcal{D}$ nodes. All Pareto optimal solutions for the problem can be obtained in $O(\mathcal{D}^4)$ time since $\mathcal{D} \leq D$.

3.2. Perfect Matching with a Bisection Search

The perfect matching with a bisection search, appeared in [15] for the bottleneck product rate variation problem with absolute-deviation objective, can also be applied for Problem F_m with necessary modifications. This procedure is more efficient than the bottleneck assignment method.

The method relies on the level curves $f_{ij}(k) = (\gamma_i|j-k\tilde{r}_i|)^m$, $j = 0,1,\dots,d_i$; $i = 0,1,\dots,n$; $k = 1,\dots,\mathcal{D}$ and the bottleneck (bound) $B > 0$. The time horizon is assumed to be continuous though is partitioned into $\mathcal{D}$ equal time-buckets i.e. $T = [1,\mathcal{D}]$. A batch (i,j) is sequenced in a time-bucket $k \in [1,\mathcal{D}]$ such that the level curves do not exceed B. This introduces the earliest sequencing time $E_m(i,j)$ and the latest sequencing time $L_m(i,j)$ for (i,j), for all i,j.

The inequalities $[\gamma_i(j-(E_m(i,j)-1)\tilde{r}_i)]^m > B$ and $[\gamma_i(j-E_m(i,j)\tilde{r}_i)]^m \leq B$,that $E_m(i,j)$ satisfies, imply $\dfrac{j-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} \leq E_m(i,j) \leq \dfrac{j-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1$.

Likewise, $L_m(i,j)$ satisfies the inequalities $[\gamma_i((L_m(i,j)-1)\tilde{r}_i-(j-1))]^m \leq B$ and $[\gamma_i(L_m(i,j)\tilde{r}_i-(j-1))]^m > B$, implying $\dfrac{j-1+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} < L_m(i,j) \leq \dfrac{j-1+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1$.

So,

Theorem 1 *For a given* B, $E_m(i,j)$ *and* $L_m(i,j)$, $i = 0,1,\ldots,n$; $j = 1,\ldots,\bar{d}_i$ *are the unique integers* $E_m(i,j) = \left\lceil \dfrac{j-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} \right\rceil$ *and* $L_m(i,j) = \left\lfloor \dfrac{j-1+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1 \right\rfloor$.

The earliest sequencing time $E_m(i,j)$ and the latest sequencing time $L_m(i,j)$ form a time window $T_m = [E_m(i,j), L_m(i,j)]$ within which (i,j), $i = 0,1,\ldots,n$; $j = 1,\ldots,\bar{d}_i$ can be sequenced with the level curves not exceeding the bottleneck.

Corollary 1 *If* (i,j) *be sequenced within the time window* $T_m = [E_m(i,j), L_m(i,j)]$, *the level curves do not exceed* B.

Proof: Let (i,j) be sequenced in time $k \in [1,D]$ such that $E_m(i,j) \leq k \leq L_m(i,j)$.

$$\Rightarrow \left\lceil \frac{j-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} \right\rceil \leq k \text{ and } k \leq \left\lfloor \frac{j-1+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1 \right\rfloor.$$

$$\Rightarrow \frac{j-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} \leq k \text{ and } k \leq \frac{j-1+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1.$$

So, $|\gamma_i(j-k\tilde{r}_i)|^m \leq B$ and $|\gamma_i((k-1)\tilde{r}_i)-(j-1))|^m \leq B$. □

Both $E_m(i,j)$ and $L_m(i,j)$ can be calculated in $O(D)$ time since $\mathcal{D} \leq D$ [15].

A V_1−convex bipartite graph $\mathcal{G} = (V_1 \cup V_2, \mathcal{E})$ is constructed sequencing (i,j) within T_m, where $V_1 = \{1,\ldots,\mathcal{D}\}$ stands for the set of sequencing time-buckets, $V_2 = \{(i,j)|i = 0,1,\ldots,n; j = 1,\ldots,\bar{d}_i$ the set of (i,j) and $\mathcal{E} = \{(k,(i,j))|k \in T_m\}$.

The earliest due date (EDD) algorithm that matches each $k \in V_1$ to the unmatched (i,j) with the smallest $L_m(i,j)$ and $(k,(i,j)) \in \mathcal{E}$ finds a perfect matching. The algorithm stops if no such (i,j) exists [15].

The sequencing times are strictly monotonic since $E_m(i,j) = \left\lceil \dfrac{j-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} \right\rceil < \left\lceil \dfrac{j+1-\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} \right\rceil = E_m(i,j+1)$ and

$L_m(i,j) = \left\lfloor \dfrac{j-1+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1 \right\rfloor < \left\lfloor \dfrac{j+\frac{1}{\gamma_i}\sqrt[m]{B}}{\tilde{r}_i} + 1 \right\rfloor = L_m(i,j+1)$ for $0 < \tilde{r}_i < 1$,

$i = 0,1\ldots,n$. This implies that the perfect matching is order-preserving.

An order-preserving perfect matching in $\mathcal{G}$ is analogous to a feasible solution to Problem F_m. It is clear that a feasible solution implies every (i,j), $i = 0,1,\ldots,n$; $j = 1,\ldots,d_j$ assigns a unique time-bucket k, $k = 1,\ldots,\mathcal{D}$ and no time-bucket remains unmatched. This creates an order-preserving perfect matching in $\mathcal{G}$. Conversely, every order-preserving perfect matching creates a bijection $(i,j) \rightarrow k$ where $(i,j) \in V_2$ and $k \in V_1$, $i = 0,1,\ldots,n$; $j = 1,\ldots,d_j$ and $k = 1,\ldots,\mathcal{D}$. Thus,

Theorem 2 *Any instance of Problem F_m has a feasible sequence if and only if, the V_1-convex bipartite graph formed by the instance has an order-preserving perfect matching.*

A perfect matching in $\mathcal{G}$ exists if and only if $|N(K)| \geq |K|$, for all K, where $N(K) = \{(i,j) : (i,j) \in V_2, \exists k \in K \, s.t. (k,(i,j)) \in \mathcal{E}\}$ and K is either an interval in V_1 or the neighborhood of an interval in V_1, [1]. This is the Hall's theorem for the existence of a perfect matching that yields a feasible solution to the problem.

Existence of a perfect matching depends on B. A perfect matching exists if B satisfies the inequalities in the following theorem. This is a certificate for the existence of a feasible solution.

Theorem 3 Problem F_m has a feasible solution if and only if, for all k_1, $k_2 \in V_1$ with $k_1 \leq k_2$ and $[E_m(i,j), L_m(i,j)] \cap [k_1, k_2] \neq \phi$, B satisfies the inequalities

$$\sum_{i=1}^{n}\left(\left\lfloor k_2\tilde{r}_i + \frac{1}{\gamma_i}\sqrt[m]{B}\right\rfloor - \left\lceil (k_1-1)\tilde{r}_i - \frac{1}{\gamma_i}\sqrt[m]{B}\right\rceil\right) \geq k_2 - k_1 + 1 \qquad \text{and}$$

$$\sum_{i=1}^{n}\left(\left\lfloor k_2\tilde{r}_i - \frac{1}{\gamma_i}\sqrt[m]{B}\right\rfloor - \left\lceil (k_1-1)\tilde{r}_i + \frac{1}{\gamma_i}\sqrt[m]{B}\right\rceil\right) \leq k_2 - k_1 + 1.$$

Proof: Let $K = [k_1, k_2] \subseteq V_1$. Then $(i,j) \in N(K)$

$$\Leftrightarrow E_m(i,j) \leq k_2 \text{ and } L_m(i,j) \geq k_1$$

$$\Leftrightarrow \frac{j - \frac{1}{\gamma_i}\sqrt[m]{B}}{r_i} \leq k_2 \text{ and } \frac{j - 1 + \frac{1}{\gamma_i}\sqrt[m]{B}}{r_i} + 1 \geq k_1$$

$$\Leftrightarrow \left\lceil (k_1-1)\tilde{r}_i + 1 - \frac{1}{\gamma_i}\sqrt[m]{B}\right\rceil \leq j \leq \left\lfloor k_2\tilde{r}_i + \frac{1}{\gamma_i}\sqrt[m]{B}\right\rfloor.$$

So, $|N(K)| \geq |K|$ if and only if

$$\sum_{i=1}^{n}\left(\left\lfloor k_2\tilde{r}_i + \frac{1}{\gamma_i}\sqrt[m]{B}\right\rfloor - \left\lceil (k_1-1)\tilde{r}_i - \frac{1}{\gamma_i}\sqrt[m]{B}\right\rceil\right) \geq k_2 - k_1 + 1$$

Let $N(K) = [k_1, k_2] \subseteq V_1$. Then, $(i, j) \in K \subseteq V_2$

$\Leftrightarrow k_1 \leq E_m(i, j)$ and $L_m(i, j) \leq k_2$

$$\Leftrightarrow k_1 \leq \frac{j - \frac{1}{\gamma_i} \sqrt[m]{B}}{\tilde{r}_i} \text{and} \frac{j - 1 + \frac{1}{\gamma_i} \sqrt[m]{B}}{\tilde{r}_i} + 1 \leq k_2$$

$$\Leftrightarrow \left\lceil (k_1 - 1)\tilde{r}_i + 1 + \frac{1}{\gamma_i} \sqrt[m]{B} \right\rceil \leq j \leq \left\lfloor k_2 \tilde{r}_i - \frac{1}{\gamma_i} \sqrt[m]{B} \right\rfloor.$$

Thus, $|N(K)| \geq |K|$ if and only if

$$\sum_{i=1}^{n} \left(\left\lfloor k_2 \tilde{r}_i - \frac{1}{\gamma_i} \sqrt[m]{B} \right\rfloor - \left\lceil (k_1 - 1)\tilde{r}_i + \frac{1}{\gamma_i} \sqrt[m]{B} \right\rceil \right) \leq k_2 - k_1 + 1. \square$$

A feasible solution with a minimum B is optimal. The minimum B can be obtained using a bisection search that runs between the lower and upper bottlenecks. A batch for some model i is sequenced at the time-bucket $k = 1$. It holds $\min(\gamma_i(1 - \tilde{r}_i))^m \leq B$. For $B = (\gamma_i(1 - \frac{1}{D}))^m$, $\left\lceil k\tilde{r}_i + \frac{1}{\gamma_i} \sqrt[m]{B} \right\rceil = \left\lceil k\tilde{r}_i + 1 - \frac{1}{D} \right\rceil \geq k\tilde{r}_i$ and $\left\lfloor k\tilde{r}_i - \frac{1}{\gamma_i} \sqrt[m]{B} \right\rfloor = \left\lfloor k\tilde{r}_i - 1 + \frac{1}{D} \right\rfloor \leq k\tilde{r}_i$, for all $k \in V_1$. The two inequalities lead to the two inequalities in Theorem 3. The lower and upper bottlenecks for Problem F_m are $(\gamma_i(1 - \tilde{r}_{max}))^m$ and $(\gamma_i(1 - \frac{1}{D}))^m$, respectively.

Theorem 4 A bisection search in the interval $[(\gamma_i(1 - \tilde{r}_{max}))^m, (\gamma_i(1 - \frac{1}{D}))^m]$ determines the minimum B in $O(logD)$ time.

Proof: Let $B = (\gamma_i(j - k\tilde{r}_i))^m$, $i = 0, 1, \ldots, n$; $j = 1, \ldots, \tilde{d}_i$ be the bottleneck for optimality.

So, $\mathcal{D}^m B = (\gamma_i \mathcal{D}j - \gamma_i k d_i)^m$ is an integer in $[(\gamma_i(\mathcal{D} - d_{max}))^m, (\gamma_i(\mathcal{D} - 1))^m]$.

Further, $\mathcal{D} \leq D$. $\square$

The time complexity to yield an optimal sequence using the bisection search is $O(DlogD)$ since $E_m(i, j)$ and $L_m(i, j)$ can be calculated in $O(D)$. The bottleneck product rate variation problem with batching can be solved with Pareto optimal solution in time $O(D^2logD)$.

Note that the time complexity can substantially be reduced when cyclic sequence exists. When cyclic sequence, consisting of u, a positive integer, subsequences with the same length, exists and is optimal [15].

Every instance has optimal sequence when the given bottleneck is the upper bottleneck. However, it is not guaranteed for smaller value.

Let $\mathcal{D} = \gcd(\tilde{d}_i, \mathcal{D})\delta_i$, $i = 1, \ldots, n$. For any feasible solution, $\left(\gamma_i \left(\left|[k\tilde{r}_i] - k\tilde{r}_i\right|\right)\right)^m \leq \left(\gamma_i \left(\left|\tilde{x}_{ik} - k\tilde{r}_i\right|\right)\right)^m$, $i = 0, 1, \ldots, n$, where $[k\tilde{r}_i]$ is the closest integer to $k\tilde{r}_i$ and $\left|[k\tilde{r}_i] - k\tilde{r}_i\right|$ is $\frac{1}{2}$ for even δ_i and $\frac{\delta_i - 1}{\delta_i}$ for odd case. It is clear that the lower bottleneck for even $\mathcal{D}$ is $\left(\frac{\gamma_i}{2}\right)^m$ and less than that for odd $\mathcal{D}$. The lower bottleneck for any $\mathcal{D}$ is $\left(\frac{\gamma_i}{2}\right)^m$.

Theorem 5 No instance $(\tilde{d}_1, \ldots, \tilde{d}_n)$, $n \geq 2$ of Problem F_m is feasible for $B < \left(\frac{\gamma_i}{2}\right)^m$.

Proof: The bottleneck $\left(\gamma_i(1 - \tilde{r}_{max})\right)^m$ implies that $1 - \tilde{r}_{max} \leq \frac{1}{\gamma_i}\sqrt[m]{E}$.

For feasible sequence, $1 - \tilde{r}_{min} \leq \frac{2}{\gamma_i}\sqrt[m]{B}$.

$\Rightarrow \sum_{i=1}^{z} \tilde{r}_i \leq \frac{z}{\gamma_i}\sqrt[m]{B}$, $\tilde{r}_i = \tilde{r}_{min}$.

$\Rightarrow \tilde{r}_{max} \leq \sum_{i=1}^{z} \tilde{r}_i \leq \frac{z}{\gamma_i}\sqrt[m]{B}$.

$\Rightarrow 1 - \tilde{r}_{max} \geq 1 - \frac{z}{\gamma_i}\sqrt[m]{B}$.

Thus, $\frac{1}{2} \leq \frac{1}{\gamma_i}\sqrt[m]{E}$. $\square$

The instances of which the copies of models do not compete for the sequencing positions have the optimal value less than $\left(\frac{\gamma_i}{2}\right)^m$. The copies of a standard instance i.e. the instance with $0 < \tilde{d}_1 \leq \ldots \leq \tilde{d}_n$, $\gcd(\tilde{d}_1, \ldots, \tilde{d}_n) = 1$, $n \geq 2$ do not compete if and only if it is power-of-two. For two model case, the optimal value is less than $\left(\frac{\gamma_i}{2}\right)^m$ if and only if the demand for one model is even and that for the other model is odd [7, 2].

4. Relation between Optimal Sequences

Any optimal sequence if exists at the bottleneck $B = 1$ of any instance for Problem F_m for some m would have been optimal for all m since all the level curves $(\gamma_i|j - k\tilde{r}_i|)^m$ meet only at $B = 1$. But the upper bottleneck $(\gamma_i(1 - \frac{1}{2}))^m$ for any Problem F_m is less than 1. This shows that for any instance there may not exist the same optimal sequence for every Problem F_m at the same bottleneck.

Theorem 6 A feasible sequence s to Problem F_1 is also feasible to Problem F_m.

Proof: Consider a feasible sequence s to problem F_1. Assume that the copy (i,j) be sequenced at the time unit k. This implies $E_1(i,j) \leq k \leq L_1(i,j)$.

For $1 \leq m < \acute{m}$, $E_m(i,j) = \left\lceil \dfrac{j - \frac{1}{2}\sqrt[m]{B}}{\gamma_i}}{r_i} \right\rceil = \left\lceil \dfrac{\frac{j - \frac{1}{2}\sqrt[m]{B}}{\gamma_i}}{r_i} + \dfrac{\sqrt[m]{B} - \sqrt[\acute{m}]{B}}{\gamma_i r_i} \right\rceil \geq E_{\acute{m}}(i,j)$

And, $L_m(i,j) = \left\lfloor \dfrac{\frac{j - 1 + \frac{1}{2}\sqrt[m]{B}}{\gamma_i}}{r_i} + 1 \right\rfloor = \left\lfloor \dfrac{\frac{j - 1 + \frac{1}{2}\sqrt[\acute{m}]{B}}{\gamma_i}}{r_i} + 1 - \dfrac{\sqrt[m]{B} - \sqrt[\acute{m}]{B}}{\gamma_i r_i} \right\rfloor \leq L_{\acute{m}}(i,j)$.

We show $E_1(i,j) \leq L_m(i,j)$.

For $B < \frac{1}{2}$, (i,j) can be sequenced in the ideal position $2^{n-i}(2j - 1)$ and

$E_1(i,j) \leq 2^{n-i}(2j - 1) \leq L_m(i,j)$.

For $\frac{1}{2} \leq B$, $E_1(i,j) = \left\lceil \dfrac{\frac{j - \frac{1}{2}B}{\gamma_i}}{r_i} \right\rceil = \left\lceil \dfrac{\frac{j - 1 + \frac{1}{2}B}{\gamma_i}}{r_i} + 1 + \dfrac{1 - \frac{4}{2}B}{\gamma_i}{r_i} \right\rceil \leq L_1(i,j)$.

This shows that $E_m(i,j) \leq \ldots \leq E_1(i,j) \leq k \leq L_1(i,j) \leq \ldots \leq L_m(i,j)$.

Further, $E_m(i,j) \leq k$

$\Rightarrow \dfrac{\frac{j - \frac{1}{2}\sqrt[m]{B}}{\gamma_i}}{r_i} \leq k$

$\Rightarrow (\gamma_i(j - k\tilde{r}_i))^m \leq B$

And $k \leq L_m(i,j)$

$\Rightarrow k \leq \dfrac{\frac{j - 1 + \frac{1}{2}\sqrt[m]{B}}{\gamma_i}}{r_i} + 1$

$\Rightarrow (\gamma_i((k - 1)\tilde{r}_i - (j - 1)))^m \leq B$.

So, $(\gamma_i |j - k\tilde{r}_i|)^m \leq B$.

Thus, s is feasible to Problem F_m. $\square$

The same does not happen in the case of optimality. The optimality of the problem differs with different objective functions.

Theorem 7 Any optimal sequence for the problem with an objective function may not be optimal in the case of different objective function.

Proof: Let s be an optimal solution of any instance $(d_1, \ldots, d_n)$ for Problem F_m with the bottleneck $(\gamma_i (1 - \tilde{r}_{max}))^m$.

The sequence s cannot even be feasible for Problem $F_{\acute{m}}$, $\acute{m} \lessdot m$ at the same bottleneck since $(1 - \tilde{r}_{max})^m \lessdot (1 - \tilde{r}_{max})^{\acute{m}}$. $\square$

Theorem 7 also shows that the converse of Theorem 6 is not true.

5. Conclusion

This paper formulates the mathematical model of the bottleneck case of the product rate variation problem with batching. A Pareto optimal solution exists. An algorithm in [11] solves the batching phase of the problem. The sequencing phase can be solved either by the bottleneck assignment method or by the perfect matching with a bisection search. The latter one is more efficient. Every instance has optimal sequence when the given bottleneck is the upper bottleneck. However, it is not guaranteed for smaller value. Cyclic sequence exists if $gcd(d_1, \ldots, d_n) \gtrdot 1$ and is optimal.

It has been shown that a feasible sequence to Problem F_1 is also feasible to Problem F_m, for all m, m being positive integer. But it does not hold for optimal sequence. Further relations between optimal sequences of the problem with different objective functions would be interesting in future research.

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