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An epidemic model for the transmission dynamics of HIV/AIDS with different clinical stages

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Abstract

In this paper, a five–dimensional mathematical model is proposed for the transmission dynamics of HIV/AIDS within a population of varying size. In writing the model, we have divided the population under consideration into five sub classes of susceptible, infective, pre-AIDS, AIDS related complex and that of AIDS patients. The model has two non- negative equilibria namely, a disease free and the endemic equilibrium. The model has been studied using stability theory. It is shown that the positive non-trivial equilibrium is always locally stable but it may become globally stable under certain condition showing that the disease becomes endemic due to constant migration of the population into the habitat. The effect of various parameters on the spread of the disease has also been discussed.

Keywords: Epidemic model, HIV/AIDS.

1. Introduction

AIDS, the acquired immunodeficiency syndrome is a fatal disease caused by a retrovirus known as the human immunodeficiency virus (HIV) which breaks down the body's immune system, leaving the victim vulnerable to a host of life-threatening opportunistic infections, neurological disorders or unusual malignancies. It causes mortality of millions of people and expenditure of enormous amount of money in healthcare and disease control. Thus, the most urgent public health problem today is to device effective strategies to minimize the destruction caused by the AIDS epidemic. It is, therefore, essential that adequate attention must be paid to the study and control of this disease.

The study is helpful in determining the demographic and economic impact of epidemic which is turn help us to develop reasonable, scientifically and socially sound intervention plans in order to reduce the spread of disease.

It is important to mathematically analyze these epidemiological models with different infectious stages so that they can be used effectively. Now we describe staged progression of HIV infection as follows.

(i) Asymptomatic carriers (Pre AIDS) – only 1%-2% of those newly infected have mononucleosis – like symptoms that may include fever, chills, aches, swollen lymphglands and an itchy rash. These symptoms disappear and there are no other symptoms for nine months or longer. Although the individual exhibits no symptoms during this stage, he or she is highly infectious. The standard HIV blood test for the presence of antibody becomes positive during this stage.

(ii) AIDS Related Complex (ARC) – The most common symptom of ARC is swollen lymphglands in the neck, armpits, or groin that persist for 3 months or more. There is severe fatigue unrelated to exercise or druguse; unexplained persistent or recurrent fevers, often with night sweats persistent cough not associated with smoking, a cold or the flu;

and persistent diarrhoea, also possible are signs of nervous system impairment, including loss of memory, inability to think clearly, loss of judgment, and/or depression.

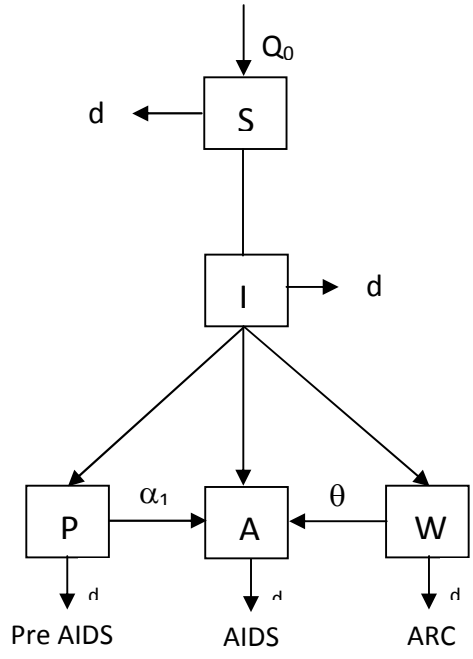
When the individual develops non-life threatening and recurrent infections such as thrush or herpes simplex, it is a signal that full-blown AIDS will occur shortly.

(iii) Full Blown AIDS – In this final stage, there is severe weight loss and weakness due to persistent diarrhoea and usually one of several opportunistic infections is present. These infections are called opportunistic because the body can usually prevent them, only an impaired immune system gives them the opportunity to get started.

A few modelling efforts have been made to study the transmission of HIV/AIDS by considering staged progression of infection to AIDS, Bailey (1986), Jacquez et al. (1989), Bailey (1989), Massad (1989), Lin (1991), Hyman et al. (1999). In view of the above in this chapter, we formulate a model that takes into different clinical stages and other demographic and epidemiological considerations.

2. Mathematical Model

Consider a population of size N(t) at time t with constant immigration rate Q_0. The Population size N(t) is divided into five subclasses of HIV negatives but susceptibles S(t), HIV positives or infectives I(t) (also assumed to be infectious), pre-AIDS P(t), AIDS related complex (ARC) W(t) and 'full blown' AIDS A(t) with natural mortality rate d. It is assumed that susceptibles become infected via sexual contact with infectives. Depending on the level of infection, a fraction $\delta_1 I$ of infectives moves to join pre-AIDS class and $\delta_2 I$ moves to develop 'full blown' AIDS directly while the remaining part of infectives i.e. $\delta_3 I$ go to ARC. It is further assumed that patients in pre-AIDS class and ARC class will ultimately develop 'full blown' AIDS as shown in the transfer diagram below,



With these considerations, the spread of disease is assumed to be governed by the following system of differential equations,

$$\frac{dS}{dt} = Q_0 - \frac{\beta_1 SI}{N} - dS \qquad S(0) = S_0 \qquad (1)$$

$$\frac{dI}{dt} = \frac{\beta_1 SI}{N} - (\delta + d)I \qquad I(0) = I_0 \qquad (2)$$

$$\frac{dP}{dt} = \delta_1 I - (\alpha_1 + d)P \qquad P(0) = P_0 \qquad (3)$$

$$\frac{dW}{dt} = \delta_3 I - (\theta + d)W \qquad W(0) = W_0 \qquad (4)$$

$$\frac{dA}{dt} = \delta_2 I + \alpha_1 P + \theta W - (\alpha + d)A \qquad A(0) = A_0 \qquad (5)$$

where Q_0 is the constant inflow of susceptibles, d is the natural mortality

rate constant assumed same in all the classes, δ is the rate of movement from infectious class, so that $(1/\delta)$ denotes the average incubation period, α is the disease-induced death rate constant and β_1 is the transmission coefficient.

Since N = S+I+P+W+A and $\delta = \delta_1+\delta_2+\delta_3$, the above equations,

It is noted that in the absence of infection, the population size approaches the steady state value Q_0/d. During the early stages of the epidemic, if it is assumed that $S \cong N \cong Q_0/d$ then the growth of infectious people I(t) can be approximately governed by the following equation,

$$\frac{dI}{dt} = [\beta_1 - (\delta + d)]I \qquad I(0) = I_0 \qquad (6)$$

which gives $I(t) = I_0 \exp[(R_0-1)/T\ ]t$ where $R_0 = \frac{\beta_1}{(\delta + d)}$, the basic reproduction rate, $T = \frac{1}{(\delta + d)}$, the time during which people remain infective and I_0 is the initial infective population at time t = 0. The doubling time t_d of the epidemic is given by

$$t_d = \frac{(\ln 2)T}{R_0 - 1} \qquad (7)$$

Thus if $R_0>1$, the infection triggers an epidemic otherwise for $R_0<1$, the epidemic dies out.

3. Stability Analysis

The model has two non-negative equilibria namely E_0 (Q_0/d, 0, 0, 0, 0), the disease free and E*(N*, I*, P*, W*, A*) the endemic equilibrium, where (N*, I*, P*, W*, A*) are positive solutions of the equations,

Solving the above equations, we get

$$N^* = \frac{1}{d}\left[Q_0 - \frac{\alpha\gamma I^*}{(\alpha+d)}\right], \quad S^* = \frac{(\delta+d)}{\beta_1}N^*$$

$$I^* = \frac{[\beta_1 - (\delta+d)]\frac{Q_0}{d}}{\frac{\alpha\gamma}{d(\alpha+d)}[\beta_1 - (\delta+d)] + \beta_1\left[1 + \frac{\delta_1}{(\alpha_1+d)} + \frac{\delta_3}{(\theta+d)} + \frac{\gamma}{(\alpha+d)}\right]}$$

$$P^* = \frac{\delta_1 I^*}{(\alpha_1+d)}, \quad W^* = \frac{\delta_3 I^*}{(\theta+d)}, \quad A^* = \frac{\gamma I^*}{(\alpha+d)}$$

where γ >0 is defined as $\gamma = \delta_2 + \frac{\alpha_1\delta_1}{(\alpha_1+d)} + \frac{\theta\delta_3}{(\theta+d)}$.

It is noted that E* is positive only when $\beta_1 > (\delta+d)$ and $Q_0 > \frac{\alpha\gamma I^*}{(\alpha+d)}$.

It is found that equilibrium level of infectives I* increases as Q_0 or β_1 increases or δ decreases leading to increase in P*, W* and A*. Further, the equilibrium level of AIDS patients A* decreases as disease-induced death rate α increases or as γ decreases and that of P* and W* increases when α_1 and θ decreases and it decreases when δ_1 and δ_3 decreases. Also equilibrium population size is a linear function of Q_0 with positive intercept. It is noted that when the disease remain endemic, the disease-induced deaths reduce the equilibrium population size from Q_0/d to N*. Now we propose a theorem for local stability of the above equilibrium points.

Theorem 1-

(i) The equilibrium point $E_0(Q_0/d, 0, 0, 0, 0)$ is locally asymptotically stable if $R_0<1$ otherwise it is unstable and then second equilibrium E*(N*, I*, P*, W*, A*) exists.

(ii) The second equilibrium E*(N*, I*, P*, W*, A*), if it exists, is locally

asymptotically stable.

Proof: To determine the local stability of E_0 and E*, we compute variational matrices $M(E_0)$and M(E*),

From $M(E_0)$, it is clear that E_0 is locally asymptotically stable (LAS) provided$\beta_1<(\delta+d)$ i.e. $R_0<1$, the disease dies out but under this condition the equilibrium E* does not exist as expected. However, if $R_0>1$ the equilibrium point E_0 is a saddle point which is stable in N-P-W-A manifold and unstable in I-direction. In such a case E* exists and the infection is maintained in the population. The characteristic equation corresponding to M(E*) is given by

$$f(\lambda)=\lambda^5+a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0$$

The conditions for local stability of the system are

$$a_1>0,\quad a_2>0,\quad a_3>0,\quad a_4>0,\quad a_5>0,\quad \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}>0,\quad \begin{vmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}>0,$$

$$\begin{vmatrix} a_1 & a_3 & a_5 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix}>0 \text{ and } \begin{vmatrix} a_1 & a_3 & a_5 & 0 & 0 \\ 1 & a_2 & a_4 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & 1 & a_2 & a_4 & 0 \\ 0 & 0 & a_1 & 0 & a_5 \end{vmatrix}>0.$$

which can be seen by some tedious calculations.

Thus the equilibrium E* is locally asymptotically stable under the condition mentioned in the theorem. We can also prove that the equilibrium E*(N*, I*, P*, W*, A*), if it exists, is globally asymptotically stable. A lemma can easily proved first.

Lemma 1- The region

$$\Omega = \left\{ (N, I, P, W, A); 0 \le N(t) \le \frac{Q_0}{d}; 0 \le I(t) \le I_{max}; 0 \le P(t) \le \frac{\delta_1 I_{max}}{(\alpha_1 + d)} \right.$$

$$\left. ; 0 \le W(t) \le \frac{\delta_3 I_{max}}{(\theta + d)},\ 0 \le A(t) \le \frac{\delta_2 I_{max} + \alpha_1 P_{max} + \theta_1 W_{max}}{(\alpha + d)} \right\}$$

is a region of attraction for $\beta_1 > (\delta + d)$, where $I_{max} = \frac{Q_0}{d}\left\{1 - \frac{(\delta + d)}{\beta_1}\right\}$.

Theorem 2- If the endemic equilibrium E* exists, then it is globally asymptotically stable provided the following conditions are satisfied in Ω,

$$\frac{\alpha_1^2}{(\alpha_1 + d)} < \frac{(\alpha + d)\delta_2}{2\delta_1},$$

$$\frac{\theta^2}{(\theta + d)} < \frac{(\alpha + d)\delta_2}{2\delta_3} \qquad (8)$$

$$\text{and } \frac{\alpha^2}{(\alpha + d)} < \frac{d^2}{4\delta_2}$$

Proof: Consider the following positive definite function about E*,

$$V = \frac{1}{2}(N - N^*)^2 + k_1\left\{I - I^* - I^* \ln\left(\frac{I}{I^*}\right)\right\} + \frac{1}{2}k_2(P - P^*)^2$$

$$+ \frac{1}{2}k_3(W - W^*)^2 + \frac{1}{2}k_4(A - A^*)^2$$

where the constant k_1, k_2, k_3 and k_4 can be chosen suitably.

The derivative of V along the solutions of the system (6.6-6.10) can be written as,

$$\frac{dV}{dt} = (N - N^*)[Q_0 - dN - \alpha A] + k_1(I - I^*)\left[\frac{\beta_1(N - I - P - W - A)}{N} - (\delta + d)\right]$$

$$+ k_2(P - P^*)[\delta_1 I - (\alpha_1 + d)P] + k_3(W - W^*)[\delta_3 I - (\theta + d)W]$$

$$+ k_4(A - A^*)[\delta_2 I + \alpha_1 P + \theta W - (\alpha + d)A]$$

After some algebraic manipulations it can be written as the sum of the quadratics as,

$$\frac{dV}{dt} = -\frac{1}{2}a_{11}(N - N^*)^2 + a_{12}(N - N^*)(I - I^*) - \frac{1}{2}a_{22}(I - I^*)^2$$

$$-\frac{1}{2}a_{11}(N - N^*)^2 + a_{15}(N - N^*)(A - A^*) - \frac{1}{2}a_{55}(A - A^*)^2$$

$$-\frac{1}{2}a_{22}(I - I^*)^2 + a_{23}(I - I^*)(P - P^*) - \frac{1}{2}a_{33}(P - P^*)^2$$

$$-\frac{1}{2}a_{22}(I - I^*)^2 + a_{24}(I - I^*)(W - W^*) - \frac{1}{2}a_{44}(W - W^*)^2$$

$$-\frac{1}{2}a_{22}(I-I^*)^2+a_{25}(I-I^*)(A-A^*)-\frac{1}{2}a_{55}(A-A^*)^2$$

$$-\frac{1}{2}a_{33}(P-P^*)^2+a_{35}(P-P^*)(A-A^*)-\frac{1}{2}a_{55}(A-A^*)^2 \qquad (9)$$

$$-\frac{1}{2}a_{44}(W-W^*)^2+a_{45}(W-W^*)(A-A^*)-\frac{1}{2}a_{55}(A-A^*)$$

where

$$a_{11}=d,\quad a_{22}=\frac{\beta_1 k_1}{2N^*},\quad a_{33}=(\alpha_1+d)k_2,\quad a_{44}=k_3(\theta+d),\quad a_{55}=\frac{k_4(\alpha+d)}{2}$$

$$a_{12}=\frac{\beta_1 k_1}{NN^*}(I+P+W+A),\quad a_{15}=-\alpha,\quad a_{23}=-\frac{\beta_1 k_1}{N^*}+k_2\delta_1,$$

$$a_{24}=-\frac{\beta_1 k_1}{N^*}+K_3\delta_3,\ a_{25}=-\frac{\beta_1 k_1}{N^*}+k_4\delta_2,\ a_{35}=k_4\alpha_1,\quad a_{45}=k_4\theta$$

Thus a sufficient condition for $\frac{dV}{dt}$ to be negative definite is that,

$$a_{12}^2-a_{11}a_{22}<0 \qquad (10i)$$

$$a_{15}^2-a_{11}a_{55}<0 \qquad (10ii)$$

$$a_{23}^2-a_{22}a_{33}<0 \qquad (10iii)$$

$$a_{24}^2-a_{22}a_{44}<0 \qquad (10iv)$$

$$a_{25}^2 - a_{22}a_{55} < 0 \qquad (10v)$$

$$a_{35}^2 - a_{33}a_{55} < 0 \qquad (10vi)$$

$$a_{45}^2 - a_{44}a_{55} < 0 \qquad (10vii)$$

Now choosing

$$k_2 = \frac{k_1\beta_1}{\delta_1 N^*}, \quad k_3 = \frac{k_1\beta_1}{\delta_3 N^*}, \quad k_4 = \frac{k_1\beta_1}{\delta_2 N^*}$$

The condition (10i-10vii) gives

$$\frac{\alpha_1^2}{(\alpha_1 + d)} < \frac{(\alpha + d)\delta_2}{2\delta_1},$$

$$\frac{\theta^2}{(\theta + d)} < \frac{(\alpha + d)\delta_2}{2\delta_3}$$

and $\dfrac{\alpha^2}{(\alpha + d)} < \dfrac{d^2}{4\delta_2}$

Where k_1 is found as $k_1 = \dfrac{dN^*}{2\beta_1}$. Hence V is a Liapunov function with respect to E* whose domain contains Ω.

4. NUMERICAL ANALYSIS AND DISCUSSION

We choose the following values of the various parameters in model (1-5) and compute the equilibrium values N*, I*, P*, W*, A*.

$Q_0 = 1333.3$, d =1/50, $\beta_1 = 1.344$, $\delta_1 = 0.3$, $\delta_2 = 0.5$, $\delta_3 = 0.2$, $\delta = 1$, $\alpha = 1$, $\alpha_1 = 0.4$, $\theta = 0.5$.

N* = 14574.17254, I* = 1159.608, P* = 839.71655, W* = 450.9588, A* = 1125.097.

Simulation is performed for initial starts I(0) = 200, P(0) = 200, W(0) = 200, A(0) = 200 as shown in figures (1-7). The variation of infective population and pre-AIDS patients, AIDS Related Complex Population and 'full blown' AIDS patient is shown for different incubation periods δ, δ_1 and δ_2 and δ_3 growth rate Q_0, transmission coefficient β_1 and disease-induced death rate α.

In fig. (1) the variation of infective population is shown with time for different values of δ i.e. rate of movement from infective class to other classes. It is seen that as the rate of movement from infective class increases, the infective population decreases. Fig. (2) shows the variation of AIDS patients with time for different values of δ_2 and it is found that with the increase in δ_2, the AIDS patients population also increases. The variation of patients is ARC with time in depicted in fig. (3) for different values of δ_3 and it is noted that as the value of δ_3 increases the ARC population also increases. Similar trend is observed for patients in pre-AIDS class as shown in fig. (4). The effect of disease-induced deaths on AIDS patients is shown in fig. (5). It is observed that as the value of α increases, the AIDS patients population decreases. Infig. (6) the role of migration is shown on AIDS patients population and it is noted that as the rate of migration increases into the community, the population of AIDS patients also increases. Fig. (7) shows the variation of AIDS population with time for different transmission coefficients and it is noted that with the increase in transmission coefficient, the AIDS population also increases.

5. Conclusion

In this chapter, a staged progression model is proposed to study the

transmission of the dreaded diseases AIDS in a population of varying size by incorporating demographic and other epidemiological considerations. It is assumed that susceptibles become infected via sexual contact with infectives and follow one of the three stages pre-AIDS, ARC and AIDS and then to death. It is also assumed that infectives in pre-AIDS and ARC will ultimately develop AIDS to join AIDS class. The model has been analyzed using stability theory and some local and global stability results are established. As usual, we have found a threshold parameter R_0 which if exceeds one, the disease persists. The model has two non-negative equilibria namely $E_0(Q_0/d, 0, 0, 0, 0)$, the disease free and $E^*(N, I^*, P^*, W^*, A^*)$, the endemic equilibrium. It is found that the equilibrium state E_0 is locally asymptotically stable if $R_0<1$ and for $R_0>1$ it is unstable and the infection is maintained in the population. The endemic equilibrium E^* which exists only when $R_0>1$ is always locally asymptotically stable. This equilibrium is also found to be globally asymptotically stable if the conditions of the theorem are satisfied. It is noted that when disease remain endemic, the disease-induced deaths reduce the equilibrium population size for Q_0/d to N^*. It is shown that equilibrium level of infectives I^* increases as Q_0 or β_1 increases or as δ decreases which leads to increase in P^*, W^* and A^*. Further, the equilibrium level of A^* decreases as disease induced death rate α increases and that of P^* and W^* increases when α_1 and θ decreases and it decreases with decrease in δ_1 and δ_3. Thus the decrease in equilibrium of infective population leads to an increase in the population of pre-AIDS, ARC and AIDS class.

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