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Convex Surface Visualization Using Rational Bi-cubic Function

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Abstract

The rational cubic function with three parameters has been extended to rational bi-cubic function to visualize the shape of regular convex surface data. The rational bi-cubic function involves six parameters in each rectangular patch. Data dependent constraints are derived on four of these parameters to visualize the shape of convex surface data while other two are free to refine the shape of surface at user choice. The developed constraints on parameters act as sufficient conditions for visualization of convex surface data. Moreover, computationally simple and less time consuming as compared to exiting techniques.

***Keywords*:** Rational bi-cubic function, convex surface, free parameters.

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1. Introduction

The data under consideration for visualization has three basic shapes, positive, monotone and convex (or concave). It is necessary to preserve these shapes to understand the meaning of underlying physical phenomenon and to transform the idea of designer to reality. The second requirement in Data Visualization environment is shape control i.e. the user can modify the shape of curve or surface without changing the data. However, this paper addresses the problem of convex surface data visualization. Convexity of data should be maintained in applications including the design of telecommunication systems, nonlinear programming arising in engineering problems [9-10], optimal control [9-10], parameter estimation [9-10] and approximation of functions [9-10]. A quadric surface paraboliod is a convex surface.

The problem of visualization of convex regular and scattered surface data has been considered by many authors. A brief review is as follows: The scheme of Asaturyan [1] divided each rectangular grid into nine sub rectangles to generate convex surfaces. This scheme is not local i.e. by changing data in the x direction of one edge of a sub rectangle there was a change throughout the grid for all sub rectangle's edges located in the original x direction. Costantini and Fontanella [6] developed a semi global scheme to preserve the shape of convex data. The drawback of the scheme was that in some rectangular patches, the degree of interpolant becomes too large and the polynomial patches tend to be linear in x and/ or y. The resulting surfaces were not always visually pleasing. Dodd, McAllister and Roulier [7] scheme preserved the convexity of the surface along the grid lines but failed to preserve the convexity in the interior of the grids and produced undesirable flat spots due to vanishing of second order mixed partial derivatives. Floater [8] derived sufficient conditions for the convexity of tensor-product Bézier surfaces. The convexity condition was generalized to C^1 tensor product B-spline surfaces. These sufficient conditions were in the form of inequalities which involved control points. The convex data visualization schemes [7-8] are failed to preserve the shape of data when data are given with derivatives. Moreover the convex data visualization schemes developed in [1, 6] are global. Hussain and Hussain [9-10] developed rational bi-cubic functions with two and four parameters to visualize regular convex surface data. Simple sufficient data dependent constraints were developed on these parameters to visualize the shape of convex surface data while no parameter was free to refine the shape of surface if required.

In this paper, we have presented an efficient convex data visualization scheme. The rational cubic function with three parameters [11] is extended to rational bi-cubic function. The rational bi-cubic function involves six parameters in each rectangular patch. The four parameters are constrained to visualize the shape of convex data while remaining two is free to modify the shape of surface at user choice. The developed scheme has many advantages over existing techniques e.g. the scheme developed in this paper is local, whereas, [1, 6] are global. The presented scheme on contrary to [7-8] is applicable to data with derivative. The schemes developed in [9-10] are automated but do not provide opportunity to the user to refine the shape of surface. The presented scheme of this paper involves two free parameters in each rectangular patch which provide opportunity to the user to refine the visual display up to his level of optimization.

The paper is organized as follows: In Section 2, rational cubic function [11] is extended to rational bi-cubic function with six parameters in each rectangular patch. In Section 3, a convex surface data visualization scheme is developed. Section 4 and 5 demonstrates and concludes the paper respectively.

2. Rational Bi-cubic Function

In this section rational cubic function developed by Sarfraz *et al.* [11] is extended to rational bi-cubic function.

Let $\left\{\left(x_i, y_j, F_{i,j}\right): i=0,1,2,\ldots,m;\ j=0,1,2,\ldots,n\right\}$ be the given data defined over the rectangular domain $D=[a,b]\times[c,d]$. The partition of the domain is $a=x_0<x_1<x_2<\ldots<x_m=b;\ c=y_0<y_1<y_2<\ldots<y_n=d$ .The local rational bi-cubic function is defined over each rectangular patch as follows:

$$S\left(x,y\right)=\frac{\left[\left(1-\theta\right)^3 G_{i,j}+\left(1-\theta\right)^2\theta H_{i,j}+\left(1-\theta\right)\theta^2 K_{i,j}+\theta^3 L_{i,j}\right]}{q_{i,j}\left(\theta\right)}, \qquad (1)$$

where

$$G_{i,j}=\frac{\alpha_{i,j}}{\hat{q}_{i,j}\left(\phi\right)}\Big[\left(1-\phi\right)^3\hat{\alpha}_{i,j}F_{i,j}+\left(1-\phi\right)^2\phi\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i,j}+\hat{\alpha}_{i,j}\hat{h}_j F^y_{i,j}\right\}$$
$$+\left(1-\phi\right)\phi^2\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i,j+1}-\hat{\beta}_{i,j}\hat{h}_j F^y_{i,j+1}\right\}+\phi^3\hat{\beta}_{i,j}F_{i,j+1}\Big],$$

$$H_{i,j}=\frac{1}{\hat{q}_{i,j}(\phi)}\Big[(1-\phi)^3\hat{\alpha}_{i,j}\left\{\left(\alpha_{i,j}+\gamma_{i,j}\right)F_{i,j}+\alpha_{i,j}h_iF^x_{i,j}\right\}+(1-\phi)^2\phi\times$$

$$\left\{\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i,j}+\hat{\alpha}_{i,j}\hat{h}_jF^y_{i,j}\right)+\alpha_{i,j}h_i\left(\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^x_{i,j}\right.\right.$$

$$\left.\left.+\hat{\alpha}_{i,j}\hat{h}_jF^{xy}_{i,j}\right)\right\}+(1-\phi)\phi^2\left\{\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i,j+1}-\hat{\beta}_{i,j}\hat{h}_jF^y_{i,j+1}\right)\right.$$

$$\left.+\alpha_{i,j}\hat{h}_j\left(\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^x_{i,j+1}-\hat{\beta}_{i,j}\hat{h}_jF^{xy}_{i,j+1}\right)\right\}+\phi^3\hat{\beta}_{i,j}\left\{\left(\alpha_{i,j}+\gamma_{i,j}\right)F_{i,j+1}\right.$$

$$\left.+\alpha_{i,j}h_iF^x_{i,j+1}\right\}\Big],$$

$$K_{i,j}=\frac{1}{\hat{q}_{i,j}(\phi)}\Big[(1-\phi)^3\hat{\alpha}_{i,j}\left\{\left(\beta_{i,j}+\gamma_{i,j}\right)F_{i+1,j}-\beta_{i,j}h_iF^x_{i+1,j}\right\}+(1-\phi)^2\phi\times$$

$$\left\{\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i+1,j}+\hat{\alpha}_{i,j}\hat{h}_jF^y_{i+1,j}\right)-\beta_{i,j}h_i\left(\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^x_{i+1,j}\right.\right.$$

$$\left.\left.+\hat{\alpha}_{i,j}\hat{h}_jF^{xy}_{i+1,j}\right)\right\}+(1-\phi)\phi^2\left\{\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i+1,j+1}\right.\right.$$

$$\left.\left.-\hat{h}_j\hat{\beta}_{i,j}F^y_{i+1,j+1}\right)-\beta_{i,j}h_i\left(\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^x_{i+1,j+1}-\hat{h}_j\hat{\beta}_{i,j}F^{xy}_{i+1,j+1}\right)\right\}$$

$$+\phi^3\hat{\beta}_{i,j}\left\{\left(\beta_{i,j}+\gamma_{i,j}\right)F_{i+1,j+1}-\beta_{i,j}h_iF^x_{i+1,j+1}\right\}\Big],$$

$$L_{i,j}=\frac{\beta_{i,j}}{\hat{q}_{i,j}(\phi)}\Big[(1-\phi)^3\hat{\alpha}_{i,j}F_{i+1,j}+(1-\phi)^2\phi\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i+1,j}+\hat{\alpha}_{i,j}h_jF^y_{i+1,j}\right\}$$

$$+(1-\phi)\phi^2\alpha_{i,j}\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i+1,j+1}-\hat{\beta}_{i,j}h_jF^y_{i+1,j+1}\right\}+\phi^3\hat{\beta}_{i,j}F_{i+1,j+1}\Big],$$

$$q_{i,j}(\theta)=\alpha_{i,j}(1-\theta)^2+\gamma_{i,j}\theta(1-\theta)+\beta_{i,j}\theta^2,$$

$$\hat{q}_{i,j}(\phi)=\hat{\alpha}_{i,j}(1-\phi)^2+\hat{\gamma}_{i,j}\phi(1-\phi)+\hat{\beta}_{i,j}\phi^2,$$

$$\theta=\frac{x-x_i}{h_i},\ \phi=\frac{y-y_j}{h_j}.$$

$\alpha_{i,j}$, $\hat{\alpha}_{i,j}$, $\beta_{i,j}$, $\hat{\beta}_{i,j}$, $\gamma_{i,j}$ and $\hat{\gamma}_{i,j}$ are the free parameters. It is worth mentioning that for $\alpha_{i,j}=\hat{\alpha}_{i,j}=\beta_{i,j}=\hat{\beta}_{i,j}=1$ and $\gamma_{i,j}=\hat{\gamma}_{i,j}=2$ the rational bi-cubic function (1) reduces to bi-cubic Hermite spline.

Zhang *et al.* [12] proposed the following result:

Lemma1. For the given interpolation function $S_{i,j}(x,y)$ the sufficient and necessary conditions for a surface $S_{i,j}(x,y)$ to be convex in rectangular grid $I_{i,j}=[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ are that for any $(x,y)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$

$$\frac{\partial^2 S_{i,j}}{\partial x^2}\cdot\frac{\partial^2 S_{i,j}}{\partial y^2}-\left(\frac{\partial^2 S_{i,j}}{\partial x\partial y}\right)^2>0$$

or $$S_{xx}S_{yy}-S_{xy}^2>0.$$

□

3. Convex Rational Bi-cubic Function

Let $(x_i,y_j,F_{i,j})$ be a convex data defined over rectangular grid $I_{i,j}=[x_i,x_{i+1}]\times[y_j,y_{j+1}];i=0,1,2,...,m-1,\ j=0,1,2,...,n-1$ such that:

$$\Delta_{i,j}\le\Delta_{i+1,j};\ F^x_{i,j}\le F^x_{i+1,j};\ F^x_{i,j}\le\Delta_{i,j}\le F^x_{i+1,j}\ ;$$
$$i=0,1,2,...,m-1;\ j=0,1,2,...,n.$$

$$\hat{\Delta}_{i,j}\le\hat{\Delta}_{i,j+1};\ F^y_{i,j}\le F^y_{i,j+1};\ F^y_{i,j}\le\hat{\Delta}_{i,j}\le F^y_{i,j+1};$$
$$i=0,1,2,...,m;\ \ j=0,1,2,...,n-1.$$

The piecewise rational bi-cubic function (1) is convex (using the Lemma 1) over each rectangular patch if

$$S_{xx}S_{yy}-S_{xy}^2>0.$$

The above equation is non linear. After changing it into liner equation (to simplify the computation)

$$S_{xx}S_{yy}-S_{xy}^2=S_{xx}S_{yy}-S_{yy}S_{xy}+S_{yy}S_{xy}-S_{xy}^2>0$$

$$\text{or } S_{yy}\left(S_{xx}-S_{xy}\right)+S_{xy}\left(S_{yy}-S_{xy}\right)>0 .$$

The above relation is equivalent to

$$S_{xx}\left(x,y\right)>0,\; S_{xy}\left(x,y\right)>0,\; S_{yy}\left(x,y\right)-S_{xy}\left(x,y\right)>0,$$

$$S_{xx}\left(x,y\right)-S_{xy}\left(x,y\right)>0,\;\; \forall\left(x,y\right)\in D . \tag{2}$$

$$S_{xx}\left(x,y\right)=\frac{\sum_{i=0}^{5}\left(1-\theta\right)^{5-i}\theta^{i}A_i}{h^{2}{}_{i}\left(q_{i,j}\left(\theta\right)\right)^{3}\hat{q}_{i,j}\left(\phi\right)}, \qquad S_{yy}\left(x,y\right)=\frac{\sum_{i=0}^{3}\left(1-\theta\right)^{3-i}\theta^{i}B_i}{\hat{h}_j{}^{2}\left(\hat{q}_{i,j}\left(\phi\right)\right)^{3}q_{i,j}\left(\theta\right)},$$

$$S_{xy}\left(x,y\right)=\frac{\sum_{i=0}^{4}\left(1-\theta\right)^{4-i}\theta^{i}C_i}{h_i\hat{h}_j\left(\hat{q}_{i,j}\left(\phi\right)\right)^{2}\left(q_{i,j}\left(\theta\right)\right)^{2}},$$

$$S_{xx}\left(x,y\right)-S_{xy}\left(x,y\right)=\frac{\sum_{i=0}^{5}\left(1-\theta\right)^{5-i}\theta^{i}D_i}{h_i\hat{h}_j\left(\hat{q}_{i,j}\left(\phi\right)\right)\left(q_{i,j}\left(\theta\right)\right)^{2}},$$

$$S_{yy}\left(x,y\right)-S_{xy}\left(x,y\right)=\frac{\sum_{i=0}^{5}\left(1-\phi\right)^{5-i}\phi^{i}E_i}{h_i\hat{h}_j\left(\hat{q}_{i,j}\left(\phi\right)\right)^{2}\left(q_{i,j}\left(\theta\right)\right)}.$$

A_i, B_i, C_i, D_i and E_i are the term involving $\alpha_{i,j}$, $\hat{\alpha}_{i,j}$, $\beta_{i,j}$, $\hat{\beta}_{i,j}$, $\gamma_{i,j}$, $\hat{\gamma}_{i,j}$ and the derivative values. The complete expressions of A_i, B_i and C_i are written as appendix. These expression easily leads to D_i and E_i.

$S_{xx}\left(x,y\right)>0$ if

$$\sum_{i=0}^{5}\left(1-\theta\right)^{5-i}\theta^{i}A_i>0,\; \left(q_{i,j}\left(\theta\right)\right)^{3}>0 \text{ and } \hat{q}_{i,j}\left(\phi\right)>0 .$$

$\left(q_{i,j}\left(\theta\right)\right)^3 > 0$ and $\hat{q}_{i,j}\left(\phi\right) > 0$ if

$$\alpha_{i,j} > 0,\ \hat{\alpha}_{i,j} > 0,\ \beta_{i,j} > 0,\ \hat{\beta}_{i,j} > 0,\ \gamma_{i,j} > 0,\ \hat{\gamma}_{i,j} > 0.$$

$\sum_{i=0}^{5}\left(1-\theta\right)^{5-i}\theta^i A_i > 0$ if

$$A_i > 0,\ i = 0,1,2,3,4,5.$$

$A_i > 0,\ i = 0,1,2,3,4,5$ if

$$\alpha_{i,j} > \left\{M_k, 1 \le k \le 4, k \in Z^+\right\},\ \beta_{i,j} > \left\{M_k, 5 \le k \le 8, k \in Z^+\right\},$$

with

$$M_1 = \frac{\gamma_{i,j}\left(F^x_{i+1,j} - \Delta_{i,j}\right)}{\left(\Delta_{i,j} - F^x_{i,j}\right)}, M_2 = \frac{\gamma_{i,j}\left(F^x_{i+1,j+1} - \Delta_{i,j+1}\right)}{\left(\Delta_{i,j+1} - F^x_{i,j+1}\right)},\ M_3 = \frac{\gamma_{i,j}\left(F^y_{i+1,j} - F^y_{i,j}\right)}{h_i F^{xy}_{i,j} - \left(F^y_{i+1,j} - F^y_{i,j}\right)},$$

$$M_4 = \frac{\gamma_{i,j}\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)}{h_i F^{xy}_{i,j+1} - \left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)}, M_5 = \frac{\gamma_{i,j}\left(\Delta_{i,j} - F^x_{i,j}\right)}{\left(F^x_{i+1,j} - \Delta_{i,j}\right)}, M_6 = \frac{\gamma_{i,j}\left(\Delta_{i,j+1} - F^x_{i,j+1}\right)}{\left(F^x_{i+1,j+1} - \Delta_{i,j+1}\right)},$$

$$M_7 = \frac{\gamma_{i,j}\left(F^y_{i+1,j} - F^y_{i,j}\right)}{h_i F^{xy}_{i+1,j} - \left(F^y_{i+1,j} - F^y_{i,j}\right)},\quad M_8 = \frac{\gamma_{i,j}\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)}{h_i F^{xy}_{i+1,j+1} - \left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)}.$$

$S_{xy}\left(x,y\right) > 0$ if

$$\sum_{i=0}^{4}\left(1-\theta\right)^{4-i}\theta^i C_i > 0.$$

$\sum_{i=0}^{4}\left(1-\theta\right)^{4-i}\theta^i C_i > 0$ if

$$C_i > 0,\ i = 0,1,2,3,4.$$

$C_i > 0$, $i = 0,1,2,3,4$ if

$$\alpha_{i,j} > \left\{ M_k, 8 \le k \le 9, k \in Z^+ \right\}, \qquad \beta_{i,j} > \left\{ M_k, 10 \le k \le 12, k \in Z^+ \right\},$$

$$\hat{\alpha}_{i,j} > \left\{ M_k, 13 \le k \le 19, k \in Z^+ \right\}, \qquad \hat{\beta}_{i,j} > \left\{ M_k, 20 \le k \le 26, k \in Z^+ \right\},$$

with

$$M_8 = \frac{\gamma_{i,j}\left(F^y_{i+1,j} - F^y_{i,j}\right)}{h_i F^{xy}_{i,j} - \left(F^y_{i+1,j} - F^y_{i,j}\right)}, M_9 = \frac{\gamma_{i,j}\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)}{h_i F^{xy}_{i,j+1} - \left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)},$$

$$M_{10} = \frac{\gamma_{i,j}\left(F^y_{i+1,j} - F^y_{i,j}\right)}{h_i F^{xy}_{i+1,j} - \left(F^y_{i+1,j} - F^y_{i,j}\right)}, \quad M_{11} = \frac{\gamma_{i,j}\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)}{h_i F^{xy}_{i+1,j+1} - \left(F^y_{i+1,j+1} - F^y_{i,j+1}\right)},$$

$$M_{12} = \frac{\gamma_{i,j}\left(F^x_{i,j+1} - F^x_{i,j}\right)}{3\left(\Delta_{i+1,j} - \Delta_{i,j}\right) - \left(F^x_{i,j+1} - F^x_{i,j}\right)}, \ M_{13} = \frac{\hat{\gamma}_{i,j}\left(F^x_{i,j+1} - F^x_{i,j}\right)}{\hat{h}_j F^{xy}_{i,j+1} - \left(F^x_{i,j+1} - F^x_{i,j}\right)},$$

$$M_{14} = \frac{\hat{\gamma}_{i,j}\left(F^x_{i+1,j+1} - F^x_{i+1,j}\right)}{\hat{h}_j F^{xy}_{i+1,j+1} - \left(F^x_{i+1,j+1} - F^x_{i+1,j}\right)}, \ M_{15} = \frac{\hat{\gamma}_{i,j}\hat{h}_j F^{xy}_{i,j+1}}{-\hat{h}_j F^{xy}_{i+1,j} + 3\left(F^x_{i,j+1} - F^x_{i,j}\right)},$$

$$M_{16} = \frac{\hat{\gamma}_{i,j}\hat{h}_j F^{xy}_{i+1,j+1}}{-\hat{h}_j F^{xy}_{i+1,j+1} + 3\left(F^x_{i,j+1} - F^x_{i,j}\right)}, \ M_{17} = \frac{\hat{\gamma}_{i,j} h_i \left(\hat{\Delta}_{i+1,j} - \hat{\Delta}_{i,j}\right)}{h_i\left(\hat{\Delta}_{i+1,j} - \hat{\Delta}_{i,j}\right) - \hat{h}_j\left(F^y_{i+1,j} - F^y_{i,j}\right)},$$

$$M_{18} = \frac{\hat{\gamma}_{i,j}\left(F^x_{i,j+1} + F^x_{i,j}\right)}{\hat{h}_j F^{xy}_{i,j} - \left(F^x_{i,j+1} + F^x_{i,j}\right)}, \quad M_{19} = \frac{\hat{\gamma}_{i,j}\left(F^x_{i+1,j+1} + F^x_{i+1,j}\right)}{\hat{h}_j F^{xy}_{i+1,j} - \left(F^x_{i+1,j+1} + F^x_{i+1,j}\right)},$$

$$M_{20}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)}{\hat{h}_j F^{xy}_{i,j+1}-\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)},\ M_{21}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)}{\hat{h}_j F^{xy}_{i+1,j+1}-\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)},$$

$$M_{22}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i,j+1}-F^{x}_{i,j}\right)}{\hat{h}_j F^{xy}_{i,j+1}-\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)},\ M_{23}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)}{\hat{h}_j F^{xy}_{i+1,j+1}-\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)},$$

$$M_{24}=\frac{\hat{\gamma}_{i,j}h_i\left(\hat{\Delta}_{i+1,j}-\hat{\Delta}_{i,j}\right)}{h_i\left(\hat{\Delta}_{i+1,j}-\hat{\Delta}_{i,j}\right)-\hat{h}_j\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},\ M_{25}=\frac{\hat{\gamma}_{i,j}\hat{h}_j F^{xy}_{i,j}}{-\hat{h}_j F^{xy}_{i,j}+3\left(F^{x}_{i,j+1}-F^{x}_{i,j}\right)},$$

$$M_{26}=\frac{\hat{\gamma}_{i,j}\hat{h}_j F^{xy}_{i+1,j}}{-\hat{h}_j F^{xy}_{i+1,j}+3\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)}.$$

Similarly, $S_{xx}(x,y)-S_{xy}(x,y)>0$ if

$$\gamma_{i,j}>0,\ \hat{\gamma}_{i,j}>0,$$

$$\alpha_{i,j}>\left\{0,M_k,27\le k\le 30,k\in Z^+\right\},\qquad \beta_{i,j}>\left\{0,M_k,31\le k\le 34,k\in Z^+\right\},$$

$$\hat{\alpha}_{i,j}>\left\{0,M_k,35\le k\le 36,k\in Z^+\right\},\qquad \hat{\beta}_{i,j}>\left\{0,M_k,37\le k\le 38,k\in Z^+\right\},$$

with

$$M_{27}=\frac{\gamma_{i,j}\left(F^{x}_{i+1,j}-\Delta_{i,j}\right)}{\left(\Delta_{i,j}-F^{x}_{i,j}\right)},\ M_{28}=\frac{\gamma_{i,j}\left(F^{x}_{i+1,j+1}-\Delta_{i,j+1}\right)}{\left(\Delta_{i,j+1}-F^{x}_{i,j+1}\right)},M_{29}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)}{h_i F^{xy}_{i,j}-\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)},$$

$$M_{30}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)}{h_i F^{xy}_{i,j+1}-\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},\ M_{31}=\frac{\gamma_{i,j}\left(\Delta_{i,j}-F^{x}_{i,j}\right)}{\left(F^{x}_{i+1,j}-\Delta_{i,j+1}\right)},M_{32}=\frac{\gamma_{i,j}\left(\Delta_{i,j+1}-F^{x}_{i,j+1}\right)}{\left(F^{x}_{i+1,j+1}-\Delta_{i,j+1}\right)},$$

$$M_{33}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)}{h_i F^{xy}_{i+1,j}-\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)},\ M_{34}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)}{h_i F^{xy}_{i+1,j+1}-\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},$$

$$M_{35}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)}{\hat{h}_jF^{xy}_{i,j}-\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)},\ M_{36}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)}{\hat{h}_jF^{xy}_{i+1,j}-\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)},$$

$$M_{37}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)}{\hat{h}_jF^{xy}_{i,j+1}-\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)},\ M_{38}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)}{\hat{h}_jF^{xy}_{i+1,j+1}-\left(F^{x}_{i+1,j+1}+F^{x}_{i+1,j}\right)}.$$

Similarly, $S_{yy}\left(x,y\right)-S_{xy}\left(x,y\right)>0$ if

$$\gamma_{i,j}>0,\ \hat{\gamma}_{i,j}>0,$$

$$\alpha_{i,j}>\left\{0,M_k,39\le k\le 42,k\in Z^{+}\right\},\qquad \beta_{i,j}>\left\{0,M_k,43\le k\le 46,k\in Z^{+}\right\},$$

$$\hat{\alpha}_{i,j}>\left\{0,M_k,47\le k\le 48,k\in Z^{+}\right\},\qquad \hat{\beta}_{i,j}>\left\{0,M_k,49\le k\le 58,k\in Z^{+}\right\},$$

with

$$M_{39}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)}{h_iF^{xy}_{i,j}-\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)},M_{40}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)}{h_iF^{xy}_{i,j+1}-\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},$$

$$M_{41}=\frac{\gamma_{i,j}h_iF^{xy}_{i+1,j}}{-h_iF^{xy}_{i+1,j}+3\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)},M_{42}=\frac{\gamma_{i,j}h_iF^{xy}_{i+1,j+1}}{-h_iF^{xy}_{i+1,j+1}+3\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},$$

$$M_{43}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)}{h_iF^{xy}_{i+1,j}-\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)},M_{44}=\frac{\gamma_{i,j}\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)}{h_iF^{xy}_{i+1,j+1}-\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},$$

$$M_{45}=\frac{\gamma_{i,j}h_iF^{xy}_{i,j}}{-h_iF^{xy}_{i,j}+3\left(F^{y}_{i+1,j}-F^{y}_{i,j}\right)},M_{46}=\frac{\gamma_{i,j}h_iF^{xy}_{i,j+1}}{-h_iF^{xy}_{i,j+1}+3\left(F^{y}_{i+1,j+1}-F^{y}_{i,j+1}\right)},$$

$$M_{47}=\frac{\hat{\gamma}_{i,j}\left(F^{y}_{i,j+1}-\hat{\Delta}_{i,j}\right)}{\left(\hat{\Delta}_{i,j}-F^{y}_{i,j}\right)},\ M_{48}=\frac{\hat{\gamma}_{i,j}\left(F^{y}_{i+1,j+1}-\hat{\Delta}_{i+1,j}\right)}{\left(\hat{\Delta}_{i+1,j}-F^{y}_{i+1,j}\right)},$$

$$M_{49}=\frac{\hat{\gamma}_{i,j}\left(F^x_{i,j+1}-F^x_{i,j}\right)}{\hat{h}_jF^{xy}_{i,j}-\left(F^x_{i,j+1}+F^x_{i,j}\right)},M_{50}=\frac{\hat{\gamma}_{i,j}\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)}{\hat{h}_jF^{xy}_{i+1,j}-\left(F^x_{i+1,j+1}+F^x_{i+1,j}\right)},$$

$$M_{51}=\frac{\hat{\gamma}_{i,j}\hat{h}_jF^{xy}_{i,j+1}}{-\hat{h}_jF^{xy}_{i,j+1}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)},M_{52}=\frac{\hat{\gamma}_{i,j}\hat{h}_jF^{xy}_{i+1,j+1}}{-\hat{h}_jF^{xy}_{i+1,j+1}+3\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)},$$

$$M_{53}=\frac{\gamma_{i,j}\left(\hat{\Delta}_{i,j}-F^y_{i,j}\right)}{\left(F^y_{i,j+1}-\hat{\Delta}_{i,j+1}\right)},\ M_{54}=\frac{\hat{\gamma}_{i,j}\left(\hat{\Delta}_{i+1,j}-F^y_{i+1,j}\right)}{\left(F^y_{i+1,j+1}-\hat{\Delta}_{i+1,j}\right)},$$

$$M_{55}=\frac{\hat{\gamma}_{i,j}\left(F^x_{i,j+1}-F^x_{i,j}\right)}{\hat{h}_jF^{xy}_{i,j+1}-\left(F^x_{i,j+1}-F^x_{i,j}\right)},\ M_{56}=\frac{\hat{\gamma}_{i,j}\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)}{\hat{h}_jF^{xy}_{i+1,j+1}-\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)},$$

$$M_{57}=\frac{\hat{\gamma}_{i,j}\hat{h}_jF^{xy}_{i+1,j}}{-\hat{h}_jF^{xy}_{i+1,j}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)},\ M_{58}=\frac{\hat{\gamma}_{i,j}\hat{h}_jF^{xy}_{i,j}}{-\hat{h}_jF^{xy}_{i,j}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)}.$$

The above discussion can be summarized as:

Theorem 1: The piecewise rational bi-cubic interpolant $S(x,y)$ defined over the rectangular mesh $D=[x_0\ x_m]\times[y_0\ y_n]$, in (1), is convex if the following sufficient conditions are satisfied:

$$\alpha_{i,j}>\left\{0,N_k,1\le k\le 6,k\in Z^+\right\},\qquad \beta_{i,j}>\left\{0,N_k,7\le k\le 13,k\in Z^+\right\},$$

$$\hat{\alpha}_{i,j}>\left\{0,N_k,14\le k\le 20,k\in Z^+\right\},\qquad \hat{\beta}_{i,j}>\left\{0,N_k,21\le k\le 27,k\in Z^+\right\},$$

where

$$N_1=\frac{\gamma_{i,j}\left(F^x_{i+1,j}-\Delta_{i,j}\right)}{\left(\Delta_{i,j}-F^x_{i,j}\right)},\ N_2=\frac{\gamma_{i,j}\left(F^x_{i+1,j+1}-\Delta_{i,j+1}\right)}{\left(\Delta_{i,j+1}-F^x_{i,j+1}\right)},$$

$$N_3=\frac{\gamma_{i,j}\left(F^y_{i+1,j}-F^y_{i,j}\right)}{h_iF^{xy}_{i,j}-\left(F^y_{i+1,j}-F^y_{i,j}\right)},N_4=\frac{\gamma_{i,j}\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)}{h_iF^{xy}_{i,j+1}-\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)},$$

$$N_5=\frac{\gamma_{i,j}h_iF^{xy}_{i+1,j}}{-h_iF^{xy}_{i+1,j}+3\left(F^y_{i+1,j}-F^y_{i,j}\right)},\ N_6=\frac{\gamma_{i,j}h_iF^{xy}_{i+1,j+1}}{-h_iF^{xy}_{i+1,j+1}+3\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)},$$

$$N_7=\frac{\gamma_{i,j}\left(\Delta_{i,j}-F^x_{i,j}\right)}{\left(F^x_{i+1,j}-\Delta_{i,j}\right)},\ N_8=\frac{\gamma_{i,j}\left(\Delta_{i,j+1}-F^x_{i,j+1}\right)}{\left(F^x_{i+1,j+1}-\Delta_{i,j+1}\right)},\ N_9=\frac{\gamma_{i,j}\left(F^y_{i+1,j}-F^y_{i,j}\right)}{h_iF^{xy}_{i+1,j}-\left(F^y_{i+1,j}-F^y_{i,j}\right)},$$

$$N_{10}=\frac{\gamma_{i,j}\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)}{h_iF^{xy}_{i+1,j+1}-\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)},\ N_{11}=\frac{\gamma_{i,j}h_iF^{xy}_{i,j}}{-h_iF^{xy}_{i,j}+3\left(F^y_{i+1,j}-F^y_{i,j}\right)},$$

$$N_{12}=\frac{\gamma_{i,j}h_iF^{xy}_{i,j+1}}{-h_iF^{xy}_{i,j+1}+3\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)},\ N_{13}=\frac{\gamma_{i,j}\left(F^x_{i,j+1}-F^x_{i,j}\right)}{3\left(\Delta_{i+1,j}-\Delta_{i,j}\right)-\left(F^x_{i,j+1}-F^x_{i,j}\right)},$$

$$N_{25}=\frac{\hat\gamma_{i,j}\hat h_jF^{xy}_{i,j}}{-\hat h_jF^{xy}_{i,j}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)},\ N_{15}=\frac{\hat\gamma_{i,j}\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)}{\hat h_jF^{xy}_{i+1,j+1}-\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)},$$

$$N_{16}=\frac{\hat\gamma_{i,j}\hat h_jF^{xy}_{i,j+1}}{-\hat h_jF^{xy}_{i+1,j}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)},\ \ N_{17}=\frac{\hat\gamma_{i,j}\hat h_jF^{xy}_{i+1,j+1}}{-\hat h_jF^{xy}_{i+1,j+1}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)},$$

$$N_{18}=\frac{\hat\gamma_{i,j}h_i\left(\hat\Delta_{i+1,j}-\hat\Delta_{i,j}\right)}{h_i\left(\hat\Delta_{i+1,j}-\hat\Delta_{i,j}\right)-\hat h_j\left(F^y_{i+1,j}-F^y_{i,j}\right)},\ N_{19}=\frac{\hat\gamma_{i,j}\left(F^x_{i,j+1}+F^x_{i,j}\right)}{\hat h_jF^{xy}_{i,j}-\left(F^x_{i,j+1}+F^x_{i,j}\right)},$$

$$N_{20}=\frac{\hat\gamma_{i,j}\left(F^x_{i+1,j+1}+F^x_{i+1,j}\right)}{\hat h_jF^{xy}_{i+1,j}-\left(F^x_{i+1,j+1}+F^x_{i+1,j}\right)},\ N_{21}=\frac{\hat\gamma_{i,j}\left(F^x_{i+1,j+1}+F^x_{i+1,j}\right)}{\hat h_jF^{xy}_{i+1,j+1}-\left(F^x_{i+1,j+1}+F^x_{i+1,j}\right)},$$

$$N_{22}=\frac{\hat\gamma_{i,j}\left(F^x_{i,j+1}-F^x_{i,j}\right)}{\hat h_jF^{xy}_{i,j+1}-\left(F^x_{i,j+1}+F^x_{i,j}\right)},\ N_{23}=\frac{\hat\gamma_{i,j}\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)}{\hat h_jF^{xy}_{i+1,j+1}-\left(F^x_{i+1,j+1}+F^x_{i+1,j}\right)},$$

$$N_{24}=\frac{\hat\gamma_{i,j}h_i\left(\hat\Delta_{i+1,j}-\hat\Delta_{i,j}\right)}{h_i\left(\hat\Delta_{i+1,j}-\hat\Delta_{i,j}\right)-\hat h_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)},N_{25}=\frac{\hat\gamma_{i,j}\hat h_jF^{xy}_{i,j}}{-\hat h_jF^{xy}_{i,j}+3\left(F^x_{i,j+1}-F^x_{i,j}\right)},$$

$$N_{26}=\frac{\hat{\gamma}_{i,j}\hat{h}_j F^{xy}_{i+1,j}}{-\hat{h}_j F^{xy}_{i+1,j}+3\left(F^{x}_{i+1,j+1}-F^{x}_{i+1,j}\right)},\ N_{27}=\frac{\hat{\gamma}_{i,j}\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)}{\hat{h}_j F^{xy}_{i,j+1}-\left(F^{x}_{i,j+1}+F^{x}_{i,j}\right)}.$$

The above constraints can be rearranged as:

$$\alpha_{i,j}=l_{i,j}+Max\left\{0,N_k,1\le k\le 6,k\in Z^+\right\};$$
$$\beta_{i,j}=m_{i,j}+Max\left\{0,N_k,7\le k\le 13,k\in Z^+\right\};$$
$$\hat{\alpha}_{i,j}=n_{i,j}+\left\{0,N_k,14\le k\le 20,k\in Z^+\right\};$$
$$\hat{\beta}_{i,j}=o_{i,j}+Max\left\{0,N_k,21\le k\le 27,k\in Z^+\right\};$$

$$l_{i,j}>0,\ m_{i,j}>0,\ n_{i,j}>0,\ o_{i,j}>0.$$

4. Numerical Examples

This section illustrates the convex surface data visualization scheme developed in Section 3.

Example 1: The convex data set presented in Table 1 is generated from the following function:

$$F_1(x,y)=x^2+y^2,\ \ -3\le x,y\le 3.$$

A convex surface in Figure 1 is produced by the convex data visualization scheme developed in Section 3. The values assigned to free parameters are $\gamma_{i,j}=5$ and $\hat{\gamma}_{i,j}=5$. Figure 2 and Figure 3 provide the *xz*-view and *yz*-view of Figure 1.

Table 1: A convex data set generated from the function $F_1(x,y)$.

y/x	-3	-2	-1	0	1	2	3
-3	18	13	10	9	10	13	18

-2	13	8	5	4	5	8	13
-1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

Example 2: The convex data set presented in Table 2 is generated from the following function:

$$F_2\left(x,y\right)=e^{\sqrt{x^2+y^2}},\ \ 1\le x,y\le 6.$$

Table 2: A convex data set of $F_2\left(x,y\right)$.

y/x	1	2	3	4	5	6
1	2	5	10	17	26	37
2	5	8	13	20	100	144
3	9	36	81	144	225	324
4	16	64	144	256	400	576
5	25	100	225	400	625	900
6	36	144	324	576	900	1296

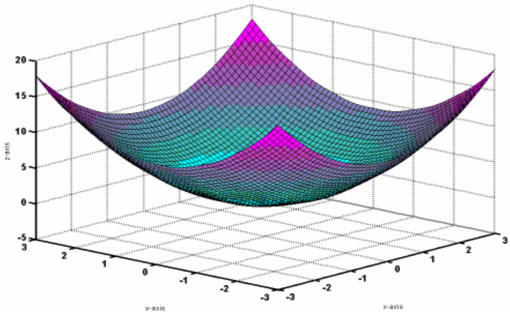

Figure 1: Convex rational bi-cubic surface.

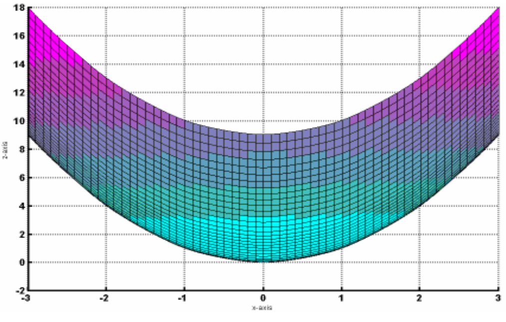

Figure 2: *xz*-view of Figure 1.

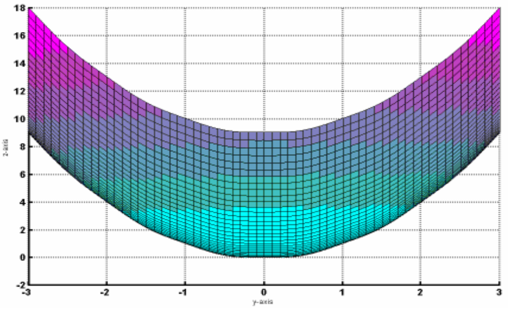

Figure 3: *yz*-view of Figure 1.

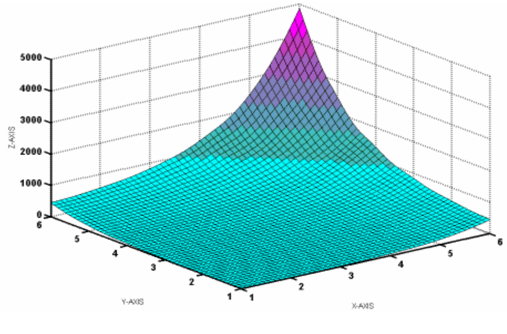

Figure 4: Convex rational bi-cubic surface.

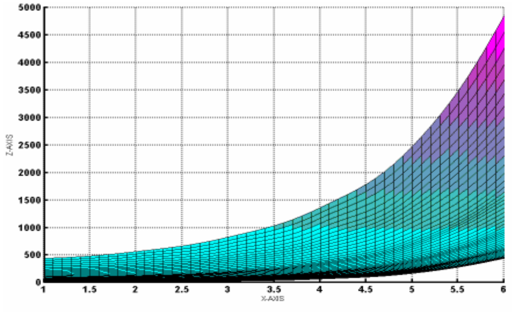

Figure 5: *xz*-view of Figure 4.

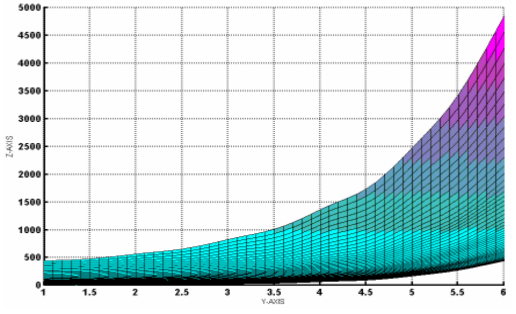

Figure 6: *yz*-view of Figure 4.

A convex surface is shown in Figure 4 that is produced by the convex surface data visualization scheme developed in Section 3 by assigning values to free parameters are $\gamma_{i,j} = 4.9$ and $\hat{\gamma}_{i,j} = 5$. Figure 5 and Figure 6 provide the *xz*-view and *yz*-view of Figure 4.

5. Conclusion

In this paper, the problem of visualization of convex surface data using rational bi-cubic function (1) is discussed. Data dependent sufficient conditions are derived on free parameters in the description of rational bi-cubic function to preserve the shape of convex data. The convexity preserving scheme developed in this paper is simple and easy to implement than the schemes developed in [1, 6-8]. The schemes developed in [1, 6] for data visualization were not local, whereas, the scheme developed in this paper is local. Thus the present method is more flexible. Further, unlike [7, 8] no derivative constraints were imposed so that the present method applies equally to data or data with derivative. However in our scheme the derivatives are approximated by arithmetic mean choice of derivatives. The presence of two free parameters in each rectangular patch makes the scheme more flexible as compared to [9-10].

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APPENDIX

$$A_0 = \sum_{j=0}^{3} (1-\phi)^{3-j} \phi^j A_{0,j}, \text{ with}$$

$$A_{0,0} = 2\alpha_{i,j}^2 \hat{\alpha}_{i,j} \left\{ \left(\beta_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j} - F_{i,j}\right) - h_i \left(\beta_{i,j} F_{i+1,j}^x + \gamma_{i,j} F_{i,j}^x\right) \right\},$$

$$A_{0,1} = \frac{\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\alpha}_{i,j}} A_{0,0} + 2\alpha_{i,j}^2 \hat{\alpha}_{i,j} \hat{h}_j \left\{ \left(\beta_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j}^y - F_{i,j}^y\right) - h_i \left(\beta_{i,j} F_{i+1,j}^{xy} + \gamma_{i,j} F_{i,j}^{xy}\right) \right\},$$

$$A_{0,2} = \frac{\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\beta}_{i,j}} A_{0,3} - 2\alpha_{i,j}^2 \hat{\beta}_{i,j} \hat{h}_j \left\{ \left(\beta_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j+1}^y - F_{i,j+1}^y\right) - h_i \left(\beta_{i,j} F_{i+1,j+1}^{xy} + \gamma_{i,j} F_{i,j+1}^{xy}\right) \right\},$$

$$A_{0,3} = 2\alpha_{i,j}^2 \hat{\beta}_{i,j} \left\{ \left(\beta_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j+1} - F_{i,j+1}\right) - h_i \left(\beta_{i,j} F_{i+1,j+1}^x + \gamma_{i,j} F_{i,j+1}^x\right) \right\}.$$

$$A_1 = \sum_{j=0}^{3} (1-\phi)^{3-j} \phi^j A_{1,j}, \text{ with}$$

$$A_{1,0} = 2\alpha_{i,j}^2 \hat{\alpha}_{i,j} \left\{ \left(5\beta_{i,j} + 2\gamma_{i,j}\right)\left(F_{i+1,j} - F_{i,j}\right) - \beta_{i,j} h_i \left(F_{i,j}^x + 2F_{i+1,j}^x\right) - \left(\beta_{i,j} + \gamma_{i,j}\right) h_i F_{i,j}^x \right\},$$

$$A_{1,1} = \frac{\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\alpha}_{i,j}} A_{1,0} + 2\alpha_{i,j}^2 \hat{\alpha}_{i,j} \hat{h}_j \left\{ \left(5\beta_{i,j} + 2\gamma_{i,j}\right)\left(F_{i+1,j}^y - F_{i,j}^y\right) - 2\beta_{i,j} h_i F_{i+1,j}^{xy} - 2\left(\beta_{i,j} + \gamma_{i,j}\right) h_i F_{i,j}^{xy} - \beta_{i,j} h_i F_{i,j}^{xy} \right\},$$

$$A_{1,2} = \frac{\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\beta}_{i,j}} A_{1,3} - 2\alpha_{i,j}^2 \hat{\beta}_{i,j} \hat{h}_j \left\{ \left(5\beta_{i,j} + 2\gamma_{i,j}\right)\left(F_{i+1,j+1}^y - F_{i,j+1}^y\right) - 2\beta_{i,j} h_i F_{i+1,j+1}^{xy} - \beta_{i,j} h_i F_{i,j+1}^{xy} - 2\left(\beta_{i,j} + \gamma_{i,j}\right) h_i F_{i,j+1}^{xy} \right\},$$

$$A_{1,3} = 2\alpha_{i,j}^2 \hat{\beta}_{i,j} \left\{ \left(5\beta_{i,j} + 2\gamma_{i,j}\right)\left(F_{i+1,j+1} - F_{i,j+1}\right) - \beta_{i,j} h_i \left(F_{i,j+1}^x + 2F_{i+1,j+1}^x\right)\right.$$

$$-2\left(\beta_{i,j}+\gamma_{i,j}\right)h_i F^x_{i,j+1}\Big\},$$

$$A_2=\sum_{j=0}^{3}\left(1-\phi\right)^{3-j}\phi^j A_{2,j}, \text{ with}$$

$$A_{2,0}=2\alpha_{i,j}\hat{\alpha}_{i,j}\Big\{\Big(\left(-3\beta_{i,j}+\alpha_{i,j}\right)+3\beta_{i,j}\left(2\alpha_{i,j}+\gamma_{i,j}\right)\Big)\left(F_{i+1,j}-F_{i,j}\right)$$
$$-5\alpha_{i,j}\beta_{i,j}h_i F^x_{i,j}\Big\}+2\alpha_{i,j}\hat{\alpha}_{i,j}h_i\Big\{-\beta_{i,j}\left(-3\beta_{i,j}+\alpha_{i,j}\right)F^x_{i+1,j}$$
$$-\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)F^x_{i,j}\Big\},$$

$$A_{2,1}=\frac{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)}{\hat{\alpha}_{i,j}}A_{2,0}+2\alpha_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\Big\{\left(\beta_{i,j}+\gamma_{i,j}\right)\left(-3\beta_{i,j}+\alpha_{i,j}\right)\left(F^y_{i+1,j}-F^y_{i,j}\right)$$
$$-5\alpha_{i,j}\beta_{i,j}h_i F^{xy}_{i,j}+3\beta_{i,j}\left(2\alpha_{i,j}+\gamma_{i,j}\right)\left(F^y_{i+1,j}-F^y_{i,j}\right)-\alpha_{i,j}h_i\left(\beta_{i,j}+\gamma_{i,j}\right)F^{xy}_{i,j}$$
$$+\beta_{i,j}h_i\left(3\beta_{i,j}-\alpha_{i,j}\right)F^{xy}_{i+1,j}\Big\},$$

$$A_{2,2}=\frac{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)}{\hat{\beta}_{i,j}}A_{2,3}-2\hat{\beta}_{i,j}\alpha_{i,j}\hat{h}_j\Big\{\left(\beta_{i,j}+\gamma_{i,j}\right)\left(-3\beta_{i,j}+\alpha_{i,j}\right)\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$
$$+3\beta_{i,j}\left(2\alpha_{i,j}+\gamma_{i,j}\right)\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)-\alpha_{i,j}h_i\left(\beta_{i,j}+\gamma_{i,j}\right)F^{xy}_{i,j+1}$$
$$-5\alpha_{i,j}\beta_{i,j}h_i F^{xy}_{i,j+1}-\beta_{i,j}h_i\left(-3\beta_{i,j}+\alpha_{i,j}\right)F^{xy}_{i+1,j+1}\Big\},$$

$$A_{2,3}=2\alpha_{i,j}\hat{\beta}_{i,j}\Big\{\Big(\left(\beta_{i,j}+\gamma_{i,j}\right)\left(-3\beta_{i,j}+\alpha_{i,j}\right)+3\beta_{i,j}\left(2\alpha_{i,j}+\gamma_{i,j}\right)\Big)\left(F_{i+1,j+1}-F_{i,j+1}\right)$$
$$-5\alpha_{i,j}\beta_{i,j}h_i F^x_{i,j+1}+h_i\Big\{-\beta_{i,j}\left(-3\beta_{i,j}+\alpha_{i,j}\right)F^x_{i+1,j}-\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)F^x_{i,j}\Big\}\Big\}.$$

$$A_3=\sum_{j=0}^{3}\left(1-\phi\right)^{3-j}\phi^j A_{3,j}, \text{ with}$$

$$A_{3,0}=2\beta_{i,j}\hat{\alpha}_{i,j}\Big\{\Big(\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(3\alpha_{i,j}-\beta_{i,j}\right)-3\alpha_{i,j}\left(2\beta_{i,j}+\gamma_{i,j}\right)\Big)\left(F_{i+1,j}-F_{i,j}\right)$$
$$-5\beta_{i,j}\hat{\alpha}_{i,j}h_i F^x_{i+1,j}+h_i\Big\{-\alpha_{i,j}\left(3\alpha_{i,j}-\beta_{i,j}\right)F^x_{i,j}+\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)F^x_{i+1,j}\Big\}\Big\},$$

$$A_{3,1}=\frac{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)}{\hat{\alpha}_{i,j}}A_{3,0}+2\beta_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\Big\{\Big(\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(3\alpha_{i,j}-\beta_{i,j}\right)-3\alpha_{i,j}\left(2\beta_{i,j}+\gamma_{i,j}\right)\Big)\times$$
$$\left(F^y_{i+1,j}-F^y_{i,j}\right)+5\alpha_{i,j}h_i F^{xy}_{i+1,j}+\beta_{i,j}h_i\left(\alpha_{i,j}+\gamma_{i,j}\right)F^{xy}_{i+1,j}-\alpha_{i,j}h_i\left(3\alpha_{i,j}-\beta_{i,j}\right)F^{xy}_{i,j}\Big\}$$

$$A_{3,2} = \frac{\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\beta}_{i,j}} A_{3,3} - 2\hat{\beta}_{i,j}\beta_{i,j}\hat{h}_j \left\{ \left(\alpha_{i,j} + \gamma_{i,j}\right)\left(3\alpha_{i,j} - \beta_{i,j}\right)\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right) \right.$$
$$-3\alpha_{i,j}\left(2\beta_{i,j} + \gamma_{i,j}\right)\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right) + 2\beta_{i,j}h_i\left(\alpha_{i,j} + \gamma_{i,j}\right)F^{xy}_{i+1,j+1}$$
$$\left. +5\beta_{i,j}h_i F^{xy}_{i+1,j+1} - \left(3\alpha_{i,j} - \beta_{i,j}\right)h_i F^{xy}_{i,j+1} \right\},$$

$$A_{3,3} = 2\beta_{i,j}\hat{\beta}_{i,j}\left\{ \left(\left(\alpha_{i,j} + \gamma_{i,j}\right)\left(3\alpha_{i,j} - \beta_{i,j}\right) - 3\alpha_{i,j}\left(2\beta_{i,j} + \gamma_{i,j}\right)\right)\left(F_{i+1,j+1} - F_{i,j+1}\right) \right.$$
$$-h_i\left\{ -\alpha_{i,j}\left(3\alpha_{i,j} - \beta_{i,j}\right)F^x_{i,j+1} + \beta_{i,j}\left(\alpha_{i,j} + \gamma_{i,j}\right)F^x_{i+1,j+1} \right\}$$
$$\left. -5\beta_{i,j}\alpha_{i,j}h_i F^x_{i+1,j+1} \right\}.$$

$$A_4 = \sum_{j=0}^{3}\left(1-\phi\right)^{3-j}\phi^j A_{4,j}, \text{ with}$$

$$A_{4,0} = 2\beta^2_{i,j}\hat{\alpha}_{i,j}\left\{ -\left(5\alpha_{i,j} + 2\gamma_{i,j}\right)\left(F_{i+1,j} - F_{i,j}\right) - \alpha_{i,j}h_i\left(2F^x_{i,j} + F^x_{i+1,j}\right) \right.$$
$$\left. +2\left(\alpha_{i,j} + \gamma_{i,j}\right)h_i F^x_{i+1,j} \right\},$$

$$A_{4,1} = \frac{\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\alpha}_{i,j}} A_{4,0} - 2\beta^2_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\left\{ \left(5\alpha_{i,j} + 2\gamma_{i,j}\right)\left(F^y_{i+1,j} - F^y_{i,j}\right) - 2\alpha_{i,j}h_i F^{xy}_{i,j} \right.$$
$$\left. -\alpha_{i,j}h_i F^{xy}_{i+1,j} - 2\left(\alpha_{i,j} + \gamma_{i,j}\right)h_i F^{xy}_{i+1,j} \right\},$$

$$A_{4,2} = \frac{\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\beta}_{i,j}} A_{4,3} + 2\beta^2_{i,j}\hat{\beta}_{i,j}\hat{h}_j\left\{ \left(5\alpha_{i,j} + 2\gamma_{i,j}\right)\left(F^y_{i+1,j+1} - F^y_{i,j+1}\right) - 2\alpha_{i,j}h_i F^{xy}_{i,j+1} \right.$$
$$\left. -\alpha_{i,j}h_i F^{xy}_{i+1,j+1} - 2\left(\alpha_{i,j} + \gamma_{i,j}\right)h_i F^{xy}_{i+1,j+1} \right\},$$

$$A_{4,3} = 2\beta^2_{i,j}\hat{\beta}_{i,j}\left\{ \left(5\alpha_{i,j} + 2\gamma_{i,j}\right)\left(-F_{i+1,j+1} + F_{i,j+1}\right) + \alpha_{i,j}h_i\left(2F^x_{i,j+1} + F^x_{i+1,j+1}\right) \right.$$
$$\left. +2\left(\alpha_{i,j} + \gamma_{i,j}\right)h_i F^x_{i+1,j+1} \right\}.$$

$$A_5 = \sum_{j=0}^{3}\left(1-\phi\right)^{3-j}\phi^j A_{5,j}, \text{ with}$$

$$A_{5,0} = 2\beta^2_{i,j}\hat{\alpha}_{i,j}\left\{ \left(\alpha_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j} - F_{i,j}\right) + h_i\left(\alpha_{i,j}F^x_{i,j} + \gamma_{i,j}F^x_{i+1,j}\right) \right\},$$

$$A_{5,1} = \frac{\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\alpha}_{i,j}} A_{5,0} - 2\beta^2_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\left\{ \left(\alpha_{i,j} + \gamma_{i,j}\right)\left(F^y_{i+1,j} - F^y_{i,j}\right) \right.$$
$$\left. -h_i\left(\alpha_{i,j}F^{xy}_{i,j} + \gamma_{i,j}F^{xy}_{i+1,j}\right) \right\},$$

$$A_{5,2} = \frac{\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)}{\hat{\beta}_{i,j}} A_{5,3} + 2\beta_{i,j}^2 \hat{\beta}_{i,j} \hat{h}_j \left\{ \left(\alpha_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j+1}^y - F_{i,j+1}^y\right) \right.$$
$$\left. - h_i \left(\alpha_{i,j} F_{i,j+1}^{xy} + \gamma_{i,j} F_{i+1,j+1}^{xy}\right) \right\},$$
$$A_{5,3} = 2\beta_{i,j}^2 \hat{\beta}_{i,j} \left\{ -\left(\alpha_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j+1} - F_{i,j+1}\right) - h_i \left(\alpha_{i,j} F_{i,j+1}^x + \gamma_{i,j} F_{i+1,j+1}^x\right) \right\}.$$
$$B_0 = \sum_{j=0}^{5} \left(1-\phi\right)^{5-j} \phi^j B_{0,j}, \text{ with}$$
$$B_{0,0} = 2\hat{\alpha}_{i,j}^2 \alpha_{i,j} \left\{ \left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i,j+1} - F_{i,j}\right) - \hat{h}_j \left(\hat{\gamma}_{i,j} F_{i,j}^y + \hat{\beta}_{i,j} F_{i,j+1}^y\right) \right\},$$
$$B_{0,1} = 2\hat{\alpha}_{i,j}^2 \alpha_{i,j} \left\{ \left(5\hat{\beta}_{i,j} + 2\hat{\gamma}_{i,j}\right)\left(F_{i,j+1} - F_{i,j}\right) - \hat{\beta}_{i,j} \hat{h}_j \left(2F_{i,j}^y + F_{i,j+1}^y\right) \right.$$
$$\left. -2\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right) \hat{h}_j F_{i,j}^y \right\},$$
$$B_{0,2} = 2\alpha_{i,j} \hat{\alpha}_{i,j} \left\{ \left(\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(-3\hat{\beta}_{i,j} + \hat{\alpha}_{i,j}\right) + 3\hat{\beta}_{i,j} \left(2\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right) \right)\left(F_{i,j+1} - F_{i,j}\right) \right.$$
$$\left. +5\hat{\alpha}_{i,j} \hat{\beta}_{i,j} \hat{h}_j F_{i,j}^y + h_i \left(\hat{\beta}_{i,j} \left(3\hat{\beta}_{i,j} - \hat{\alpha}_{i,j}\right) F_{i,j+1}^y - \hat{\alpha}_{i,j} \left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right) F_{i,j}^y \right) \right\},$$
$$B_{0,3} = 2\alpha_{i,j} \hat{\beta}_{i,j} \left\{ \left(\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left(-\hat{\beta}_{i,j} + 3\hat{\alpha}_{i,j}\right) - 3\hat{\alpha}_{i,j} \left(2\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right) \right)\left(F_{i,j+1} - F_{i,j}\right) \right.$$
$$\left. +5\hat{\alpha}_{i,j} \hat{\beta}_{i,j} \hat{h}_j F_{i,j+1}^y + \hat{\alpha}_{i,j} \hat{h}_j \left(\left(\hat{\beta}_{i,j} - 3\hat{\alpha}_{i,j}\right) F_{i,j}^y + \left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right) F_{i,j+1}^y \right) \right\},$$
$$B_{0,4} = 2\hat{\beta}_{i,j}^2 \alpha_{i,j} \left\{ \left(5\hat{\alpha}_{i,j} + 2\hat{\gamma}_{i,j}\right)\left(F_{i,j+1} - F_{i,j}\right) - \hat{\alpha}_{i,j} \hat{h}_j \left(2F_{i,j}^y + F_{i,j+1}^y\right) \right.$$
$$\left. +2\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right) \hat{h}_j F_{i,j+1}^y \right\},$$
$$B_{0,5} = 2\hat{\beta}_{i,j}^2 \alpha_{i,j} \left\{ -\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i,j+1} - F_{i,j}\right) + \hat{h}_j \left(\hat{\alpha}_{i,j} F_{i,j}^y + \hat{\gamma}_{i,j} F_{i,j+1}^y\right) \right\},$$
$$B_1 = \sum_{j=0}^{5} \left(1-\phi\right)^{5-j} \phi^j B_{1,j}, \text{ with}$$
$$B_{1,0} = \frac{\left(\alpha_{i,j} + \gamma_{i,j}\right)}{\alpha_{i,j}} B_{0,0} + 2\hat{\alpha}_{i,j}^2 \alpha_{i,j} h_i \left\{ \left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i,j+1}^x - F_{i,j}^x\right) \right.$$
$$\left. -\hat{h}_j \left(\hat{\beta}_{i,j} F_{i,j+1}^{xy} + \hat{\gamma}_{i,j} F_{i,j}^{xy}\right) \right\},$$
$$B_{1,1} = \frac{\left(\alpha_{i,j} + \gamma_{i,j}\right)}{\alpha_{i,j}} B_{0,1} + 2\hat{\alpha}_{i,j}^2 \alpha_{i,j} h_i \left\{ \left(5\hat{\beta}_{i,j} + 2\hat{\gamma}_{i,j}\right)\left(F_{i,j+1}^x - F_{i,j}^x\right) - 2\beta_{i,j} \hat{h}_j F_{i+1,j}^{xy} \right.$$
$$\left. -2\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right) \hat{h}_j F_{i,j}^{xy} - \hat{\beta}_{i,j} \hat{h}_j F_{i,j}^{xy} \right\},$$

$$B_{1,2}=\frac{\left(\alpha_{i,j}+\gamma_{i,j}\right)}{\alpha_{i,j}}B_{0,2}+2\alpha_{i,j}\hat{\alpha}_{i,j}h_i\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(-3\hat{\beta}_{i,j}+\hat{\alpha}_{i,j}\right)\left(F^x_{i,j+1}-F^x_{i,j}\right)\right.$$

$$-5\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\hat{h}_jF^{xy}_{i,j}+3\hat{\beta}_{i,j}\left(2\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i,j+1}-F^x_{i,j}\right)-\hat{\alpha}_{i,j}\hat{h}_j\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i,j}$$

$$\left.+2\hat{\beta}_{i,j}\hat{h}_j\left(3\hat{\beta}_{i,j}-\hat{\alpha}_{i,j}\right)F^{xy}_{i,j+1}\right\},$$

$$B_{1,3}=\frac{\left(\alpha_{i,j}+\gamma_{i,j}\right)}{\alpha_{i,j}}B_{0,3}+2\alpha_{i,j}\hat{\beta}_{i,j}h_i\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(-\hat{\beta}_{i,j}+3\hat{\alpha}_{i,j}\right)\left(F^x_{i,j+1}-F^x_{i,j}\right)\right.$$

$$+\hat{\beta}_{i,j}\hat{h}_j\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i,j+1}-6\hat{\alpha}_{i,j}\left(2\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i,j+1}-F^x_{i,j}\right)$$

$$\left.+5\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\hat{h}_jF^{xy}_{i,j+1}+\hat{\alpha}_{i,j}\hat{h}_j\left(\hat{\beta}_{i,j}-3\hat{\alpha}_{i,j}\right)F^{xy}_{i,j}\right\},$$

$$B_{1,4}=\frac{\left(\alpha_{i,j}+\gamma_{i,j}\right)}{\alpha_{i,j}}B_{0,4}+2\hat{\beta}^2_{i,j}\alpha_{i,j}h_i\left\{\left(5\hat{\alpha}_{i,j}+2\hat{\gamma}_{i,j}\right)\left(F^x_{i,j+1}-F^x_{i,j}\right)+\hat{\alpha}_{i,j}\hat{h}_jF^{xy}_{i,j+1}\right.$$

$$\left.+2\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_jF^{xy}_{i,j+1}+2\hat{\alpha}_{i,j}\hat{h}_jF^{xy}_{i,j}\right\},$$

$$B_{1,5}=\frac{\left(\alpha_{i,j}+\gamma_{i,j}\right)}{\alpha_{i,j}}B_{0,5}-2\hat{\beta}^2_{i,j}\alpha_{i,j}h_i\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i,j+1}-F^x_{i,j}\right)\right.$$

$$\left.-\hat{h}_j\left(\hat{\alpha}_{i,j}F^{xy}_{i,j}+\hat{\gamma}_{i,j}F^{xy}_{i,j+1}\right)\right\},$$

$$B_2=\sum_{j=0}^{5}\left(1-\phi\right)^{5-j}\phi^jB_{2,j},\ \text{with}$$

$$B_{2,0}=\frac{\left(\beta_{i,j}+\gamma_{i,j}\right)}{\beta_{i,j}}B_{3,0}-2\hat{\alpha}^2_{i,j}\beta_{i,j}h_i\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)\right.$$

$$\left.-\hat{h}_j\left(\hat{\beta}_{i,j}F^{xy}_{i+1,j+1}+\hat{\gamma}_{i,j}F^{xy}_{i+1,j}\right)\right\},$$

$$B_{2,1}=\frac{\left(\beta_{i,j}+\gamma_{i,j}\right)}{\beta_{i,j}}B_{3,1}+2\hat{\alpha}^2_{i,j}\beta_{i,j}h_i\left\{\left(5\hat{\beta}_{i,j}+2\hat{\gamma}_{i,j}\right)\left(-F^x_{i+1,j+1}+F^x_{i+1,j}\right)+2\hat{\beta}_{i,j}\hat{h}_jF^{xy}_{i+1,j+1}\right.$$

$$\left.+\hat{\beta}_{i,j}\hat{h}_jF^{xy}_{i+1,j}+2\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_jF^{xy}_{i+1,j}\right\},$$

$$B_{2,2}=\frac{\left(\beta_{i,j}+\gamma_{i,j}\right)}{\beta_{i,j}}B_{3,2}+2\beta_{i,j}\hat{\alpha}_{i,j}h_i\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(3\hat{\beta}_{i,j}-\hat{\alpha}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)\right.$$

$$-3\hat{\beta}_{i,j}\left(2\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)+\hat{\alpha}_{i,j}\hat{h}_j\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i+1,j}$$

$$\left.+5\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\hat{h}_jF^{xy}_{i+1,j}\right\}+2\alpha_{i,j}\hat{\beta}_{i,j}\beta_{i,j}h_i\hat{h}_j\left(-3\hat{\beta}_{i,j}+\hat{\alpha}_{i,j}\right)F^{xy}_{i+1,j+1},$$

$$B_{2,3}=\frac{\left(\beta_{i,j}+\gamma_{i,j}\right)}{\beta_{i,j}}B_{3,3}+2\beta_{i,j}\hat{\beta}_{i,j}h_i\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\hat{\beta}_{i,j}-3\hat{\alpha}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)\right.$$

$$+3\hat{\alpha}_{i,j}\left(2\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)-\hat{\beta}_{i,j}\hat{h}_j\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i+1,j+1}$$

$$\left.-5\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\hat{h}_jF^{xy}_{i+1,j+1}+\hat{\alpha}_{i,j}\hat{h}_j\left(-\hat{\beta}_{i,j}+3\hat{\alpha}_{i,j}\right)F^{xy}_{i+1,j}\right\},$$

$$B_{2,4}=\frac{\left(\beta_{i,j}+\gamma_{i,j}\right)}{\beta_{i,j}}B_{3,4}+2\hat{\beta}^2_{i,j}\beta_{i,j}h_i\left\{\left(5\hat{\alpha}_{i,j}+2\hat{\gamma}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)\right.$$

$$\left.-2\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\left(F^{xy}_{i+1,j}+F^{xy}_{i+1,j+1}\right)-4\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_jF^{xy}_{i+1,j+1}\right\},$$

$$B_{2,5}=\frac{\left(\beta_{i,j}+\gamma_{i,j}\right)}{\beta_{i,j}}B_{3,5}+2\hat{\beta}^2_{i,j}\beta_{i,j}h_i\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)\right.$$

$$\left.-\hat{h}_j\left(\hat{\alpha}_{i,j}F^{xy}_{i+1,j}+\hat{\gamma}_{i,j}F^{xy}_{i+1,j+1}\right)\right\},$$

$$B_3=\sum_{j=0}^{5}\left(1-\phi\right)^{5-j}\phi^jB_{3,j},\ \text{with}$$

$$B_{3,0}=2\hat{\alpha}^2_{i,j}\hat{\beta}_{i,j}\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)-\hat{h}_j\left(\hat{\beta}_{i,j}F^y_{i+1,j+1}+\hat{\gamma}_{i,j}F^y_{i,j+1}\right)\right\},$$

$$B_{3,1}=2\hat{\alpha}^2_{i,j}\beta_{i,j}\left\{\left(5\hat{\beta}_{i,j}+2\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}-F_{i+1,j}\right)-\hat{\beta}_{i,j}\hat{h}_j\left(F^y_{i+1,j}+2F^y_{i+1,j+1}\right)\right\}$$

$$-4\hat{\beta}^2_{i,j}\beta_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_jF^y_{i+1,j+1},$$

$$B_{3,2}=2\hat{\alpha}_{i,j}\beta_{i,j}\left\{\left(\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(-3\hat{\beta}_{i,j}+\hat{\alpha}_{i,j}\right)+3\hat{\beta}_{i,j}\left(2\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)\right.$$

$$\left.-5\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\hat{h}_jF^y_{i+1,j}-\hat{h}_j\left\{\hat{\beta}_{i,j}\left(3\hat{\beta}_{i,j}-\hat{\alpha}_{i,j}\right)F^y_{i+1,j+1}-\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^y_{i+1,j}\right\}\right\},$$

$$B_{3,3}=2\hat{\beta}_{i,j}\beta_{i,j}\left\{\left(\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(-\hat{\beta}_{i,j}+3\hat{\alpha}_{i,j}\right)-3\hat{\alpha}_{i,j}\left(2\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)\right.$$

$$\left.+5\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\hat{h}_jF^y_{i+1,j+1}+\hat{h}_j\left\{\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}-3\hat{\alpha}_{i,j}\right)F^y_{i+1,j}+\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^y_{i+1,j+1}\right\}\right\},$$

$$B_{3,4}=2\hat{\beta}^2_{i,j}\beta_{i,j}\left\{\left(5\hat{\alpha}_{i,j}+2\hat{\gamma}_{i,j}\right)\left(-F_{i+1,j+1}+F_{i+1,j}\right)+\hat{\alpha}_{i,j}\hat{h}_j\left(2F^y_{i+1,j}+F^y_{i+1,j+1}\right)\right.$$

$$\left.+2\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_jF^y_{i+1,j+1}\right\},$$

$$B_{3,5}=2\hat{\beta}^2_{i,j}\beta_{i,j}\left\{-\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)+\hat{h}_j\left(\hat{\alpha}_{i,j}F^y_{i+1,j}+\hat{\gamma}_{i,j}F^y_{i+1,j+1}\right)\right\}.$$

$$C_0=\sum_{j=0}^{4}\left(1-\phi\right)^{4-j}\phi^jC_{0,j},\ \text{with}$$

$$C_{0,0}=\alpha^2_{i,j}\hat{\alpha}^2_{i,j}\hat{h}_jh_iF^{xy}_{i,j},$$

$$C_{0,1} = 2\hat{\alpha}_{i,j}\alpha_{i,j}^2 h_i \left\{ \left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i,j+1}^x - F_{i,j}^x\right) - \hat{\beta}_{i,j}\hat{h}_j F_{i,j+1}^{xy} \right\},$$

$$C_{0,2} = 2\hat{\beta}_{i,j}\alpha_{i,j}^2 h_i \left\{ \left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i,j+1}^x - F_{i,j}^x\right) - \hat{\alpha}_{i,j}\hat{h}_j F_{i,j}^{xy} \right\},$$

$$C_{0,3} = \alpha_{i,j}^2 h_i \Big\{ 3\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\left(F_{i,j+1}^x - F_{i,j}^x\right) + \left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i,j+1}^x + F_{i,j}^x\right)$$
$$+ \hat{h}_j \left(\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right) F_{i,j}^{xy} - \hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right) F_{i,j+1}^{xy} \right) \Big\},$$

$$C_{0,4} = \alpha_{i,j}^2 \hat{\beta}_{i,j}^2 \hat{h}_j h_i F_{i,j+1}^{xy}.$$

$$C_1 = \sum_{j=0}^{4} (1-\phi)^{4-j} \phi^j C_{1,j}, \text{ with}$$

$$C_{1,0} = -2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} C_{4,0} + 2\alpha_{i,j}\hat{\alpha}_{i,j}^2 \left(\beta_{i,j} + \gamma_{i,j}\right)\hat{h}_j \left(F_{i+1,j}^y - F_{i,j}^y\right),$$

$$C_{1,1} = -2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} C_{4,1} - 4\alpha_{i,j}\hat{\alpha}_{i,j} \Big\{ \hat{\beta}_{i,j}\left(\beta_{i,j} + \gamma_{i,j}\right)\hat{h}_j \left(F_{i+1,j+1}^y - F_{i,j+1}^y\right)$$
$$\left(\beta_{i,j} + \gamma_{i,j}\right)\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1} - F_{i,j+1}\right) + \left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(\beta_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j} - F_{i,j}\right) \Big\},$$

$$C_{1,2} = -2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} C_{4,2} - 4\alpha_{i,j}\hat{\beta}_{i,j} \Big\{ \hat{\alpha}_{i,j}\left(\beta_{i,j} + \gamma_{i,j}\right)\hat{h}_j \left(F_{i+1,j}^y - F_{i,j}^y\right)$$
$$- \left(\beta_{i,j} + \gamma_{i,j}\right)\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left\{F_{i+1,j} - F_{i,j} + F_{i+1,j+1} - F_{i,j+1}\right\} \Big\},$$

$$C_{1,3} = -2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} C_{4,3} + 6\alpha_{i,j}\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\left(\beta_{i,j} + \gamma_{i,j}\right)\left(F_{i+1,j+1} - F_{i+1,j} - F_{i,j+1} + F_{i,j}\right)$$
$$+ 2\alpha_{i,j}\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left(\beta_{i,j} + \gamma_{i,j}\right)\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1} - F_{i,j}\right)$$
$$- 2\alpha_{i,j}\hat{\beta}_{i,j}\left(\beta_{i,j} + \gamma_{i,j}\right)\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\hat{h}_j \left(F_{i+1,j+1}^y - F_{i,j+1}^y\right)$$
$$+ 2\alpha_{i,j}\hat{\alpha}_{i,j}\left(\beta_{i,j} + \gamma_{i,j}\right)\left(\hat{\beta}_{i,j} + \hat{\gamma}_{i,j}\right)\hat{h}_j \left(F_{i+1,j}^y - F_{i,j}^y\right),$$

$$C_{1,4} = -2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}} C_{4,4} + 2\alpha_{i,j}\hat{\beta}_{i,j}^2 \left(\beta_{i,j} + \gamma_{i,j}\right)\hat{h}_j \left(F_{i+1,j+1}^y - F_{i,j+1}^y\right).$$

$$C_2 = \sum_{j=0}^{4} (1-\phi)^{4-j} \phi^j C_{2,j}, \text{ with}$$

$$C_{2,0} = \hat{\alpha}_{i,j}\gamma_{i,j}\left(\hat{\alpha}_{i,j} + \hat{\gamma}_{i,j}\right)\left(\beta_{i,j} + \gamma_{i,j}\right) h_i F_{i,j}^x + \hat{\alpha}_{i,j}^2 \left(\alpha_{i,j} + \gamma_{i,j}\right)\left(\beta_{i,j} + \gamma_{i,j}\right)\hat{h}_j \left(F_{i+1,j}^y - F_{i,j}^y\right)$$
$$+ 3\hat{\alpha}_{i,j}^2 \alpha_{i,j}\beta_{i,j}\hat{h}_j - \hat{\alpha}_{i,j}^2 \hat{h}_j h_i \left(\alpha_{i,j}\left(\beta_{i,j} + \gamma_{i,j}\right) F_{i,j}^{xy} + \beta_{i,j}\left(\alpha_{i,j} + \gamma_{i,j}\right) F_{i+1,j}^{xy} \right),$$

$$C_{2,1}=6\alpha_{i,j}\beta_{i,j}\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)-\Big\{6\alpha_{i,j}\beta_{i,j}\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)$$
$$-2\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\Big\}\left(F_{i+1,j}-F_{i,j}\right)$$
$$+2\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)$$
$$-2\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$
$$-2\hat{\alpha}_{i,j}\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)h_i\left(F^x_{i,j+1}-F^x_{i,j}\right)$$
$$-6\beta_{i,j}\hat{\beta}_{i,j}\alpha_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)+2\hat{\alpha}_{i,j}\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\times$$
$$\hat{\beta}_{i,j}h_i\hat{h}_jF^{xy}_{i+1,j+1}+2\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)h_i\hat{h}_jF^{xy}_{i,j+1},$$

$$C_{2,2}=\Big\{6\alpha_{i,j}\beta_{i,j}\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)-2\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\Big\}\times$$
$$\left(F_{i+1,j+1}-F_{i,j+1}\right)-2\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j}-F_{i,j}\right)$$
$$-2\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$
$$-6\beta_{i,j}\hat{\beta}_{i,j}\alpha_{i,j}\hat{\alpha}_{i,j}\hat{h}_j\left(F^y_{i+1,j}-F^y_{i,j}\right)-2\hat{\beta}_{i,j}\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)h_i\times$$
$$\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)-2\hat{\beta}_{i,j}\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)h_i\left(F^x_{i,j+1}-F^x_{i,j}\right)$$
$$+2\hat{\alpha}_{i,j}\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{\beta}_{i,j}h_i\hat{h}_jF^{xy}_{i+1,j}+2\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)h_i\hat{h}_jF^{xy}_{i,j},$$

$$C_{2,3}=-9\alpha_{i,j}\hat{\alpha}_{i,j}\beta_{i,j}\hat{\beta}_{i,j}\left(F_{i+1,j}-F_{i,j}\right)+9\alpha_{i,j}\hat{\alpha}_{i,j}\beta_{i,j}\hat{\beta}_{i,j}\left(F_{i+1,j+1}-F_{i,j+1}\right)$$
$$+3\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\left(F_{i+1,j}-F_{i,j}\right)+\Big\{3\hat{\alpha}_{i,j}\hat{\beta}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)$$
$$\times\left(\beta_{i,j}+\gamma_{i,j}\right)+3\alpha_{i,j}\beta_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\Big\}\left(F_{i+1,j+1}-F_{i,j+1}\right)$$
$$\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\left(F_{i+1,j}-F_{i,j}\right)+\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\times$$
$$\left(\beta_{i,j}+\gamma_{i,j}\right)\left(F_{i+1,j+1}-F_{i,j+1}\right)-3\beta_{i,j}\hat{\beta}_{i,j}\alpha_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$
$$+\hat{\alpha}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j}-F^y_{i,j}\right)$$
$$-\hat{\beta}_{i,j}\hat{h}_j\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$
$$+3\alpha_{i,j}\hat{\alpha}_{i,j}\beta_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j}-F^y_{i,j}\right)-3\beta_{i,j}\hat{\beta}_{i,j}\hat{\alpha}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)h_i\times$$
$$\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)-3\alpha_{i,j}\hat{\alpha}_{i,j}\beta_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)h_i\left(F^x_{i,j+1}-F^x_{i,j}\right)$$
$$-\alpha_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)h_i\left(\beta_{i,j}+\gamma_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i,j+1}+F^x_{i,j}\right)$$
$$-\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)h_i\left(F^x_{i+1,j+1}+F^y_{i+1,j}\right)$$

$$+\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)h_i\hat{h}_j\left\{\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i,j+1}-\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i,j}\right\}$$

$$+\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{h}_jh_i\left\{\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i+1,j+1}-\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F^{xy}_{i+1,j}\right\},$$

$$C_{2,4}=\hat{\beta}^2_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\beta_{i,j}+\gamma_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$

$$-\hat{\beta}^2_{i,j}\hat{h}_jh_i\left(\alpha_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)F^{xy}_{i,j+1}+\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)F^{xy}_{i+1,j+1}\right)$$

$$+3\hat{\beta}^2_{i,j}\alpha_{i,j}\beta_{i,j}\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$

$C_3=\sum_{j=0}^{4}\left(1-\phi\right)^{4-j}\phi^jC_{3,j}$, with

$$C_{3,0}=-2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}}C_{1,0}+2\beta_{i,j}\hat{\alpha}^2_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j}-F^y_{i,j}\right),$$

$$C_{3,1}=-4\beta_{i,j}\hat{\alpha}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left\{\hat{\beta}_{i,j}\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)-\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}-F_{i,j+1}\right.\right.$$

$$\left.\left.-F_{i+1,j}+F_{i,j}\right)\right\}-2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}}C_{1,1},$$

$$C_{3,2}=-2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}}C_{1,2}-4\beta_{i,j}\hat{\beta}_{i,j}\left\{\hat{\alpha}_{i,j}\left(\beta_{i,j}+\gamma_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j}-F^y_{i,j}\right)+\left(\alpha_{i,j}+\gamma_{i,j}\right)\times\right.$$

$$\left.\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j}-F_{i,j}-F_{i+1,j+1}+F_{i,j+1}\right)\right\},$$

$$C_{3,3}=-2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}}C_{1,3}+6\hat{\alpha}_{i,j}\beta_{i,j}\hat{\beta}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(F_{i+1,j+1}-F_{i+1,j}\right)-6\hat{\alpha}_{i,j}\beta_{i,j}\hat{\beta}_{i,j}\times$$

$$\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(F_{i,j+1}-F_{i,j}\right)+2\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\times$$

$$\left(F_{i+1,j+1}-F_{i,j+1}\right)+2\beta_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j}-F_{i,j}\right)$$

$$-2\beta_{i,j}\hat{\beta}_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right)$$

$$+2\hat{\alpha}_{i,j}\beta_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j}-F^y_{i,j}\right),$$

$$C_{3,4}=-2\frac{\hat{\alpha}_{i,j}}{\hat{\beta}_{i,j}}C_{1,4}+2\beta_{i,j}\hat{\beta}^2_{i,j}\left(\alpha_{i,j}+\gamma_{i,j}\right)\hat{h}_j\left(F^y_{i+1,j+1}-F^y_{i,j+1}\right).$$

$C_4=\sum_{j=0}^{4}\left(1-\phi\right)^{4-j}\phi^jC_{4,j}$, with

$$C_{4,0}=\beta^2_{i,j}\hat{\alpha}^2_{i,j}\hat{h}_jh_iF^{xy}_{i+1,j},$$

$$C_{4,1}=2\hat{\alpha}_{i,j}\beta^2_{i,j}h_i\left\{\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F^x_{i+1,j+1}-F^x_{i+1,j}\right)-\hat{\beta}_{i,j}\hat{h}_jF^{xy}_{i+1,j+1}\right\},$$

$$C_{4,2} = 2\hat{\beta}_{i,j}\beta_{i,j}^{2}h_i\left\{\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)\left(F_{i+1,j+1}^{x}-F_{i+1,j}^{x}\right)-\hat{\alpha}_{i,j}\hat{h}_jF_{i+1,j}^{xy}\right\},$$

$$C_{4,3} = 3\hat{\alpha}_{i,j}\beta_{i,j}^{2}\hat{\beta}_{i,j}h_i\left(F_{i+1,j+1}^{x}-F_{i+1,j}^{x}\right)+\beta_{i,j}^{2}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)h_i\left(F_{i+,j+1}^{x}+F_{i+1,j}^{x}\right)$$
$$+\beta_{i,j}^{2}\hat{h}_jh_i\left(\hat{\alpha}_{i,j}\left(\hat{\beta}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i+1,j}^{xy}-\hat{\beta}_{i,j}\left(\hat{\alpha}_{i,j}+\hat{\gamma}_{i,j}\right)F_{i+1,j+1}^{xy}\right),$$

$$C_{4,4} = \beta_{i,j}^{2}\hat{\beta}_{i,j}^{2}\hat{h}_jh_iF_{i+1,j+1}^{xy}.$$