



## **Some Results on facets for linear inequality in 0-1 variables**

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### **Abstract**

The facet of Knapsack ploytope, i.e. convex hull of 0-1 points satisfying a given linear inequality has been presented in this current paper. Such type of facets plays an important role in set covering set partitioning, matroidal-intersection vertex- packing, generalized assignment and other combinatorial problems. Strong covers for facets of Knapsack ploytope has been developed in the first part of the present paper. Generating family of valid cutting planes that satisfy inequality with 0-1 variables through algorithms are the attraction of this paper.

**Keywords:** Convex- hull, set-covering, set-partitioning, Matroidal-intersection, vertex-packing, cutting-planes.

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### 1. Introduction

The facets of Knapsack polytope, i.e. of convex –hull of 0-1 points satisfying a given linear inequality play an important role in set covering, set partitioning, matroidal intersection, vertex packing, generalized assignment and other combinatorial problems. It is known that every lower dimensional facet can be augmented to give rise to one or more facets of full dimension. Every strong cover gives rise to at least one different facet. A necessary and sufficient condition for an inequality with 0-1 coefficients to be a facet is reviewed.

### 2. Definitions and Notation

Consider the inequality.

$$\sum_{j=1}^n a_j \times_j \leq a_o \tag{1}$$

Where  $a_0 > a_j > 0$ , and  $x_j = 0, 1$ .

**Definition 1:** Let  $N = \{1,2,\dots,\dots,\dots, n\}$ . Let  $S$  be a subset of  $N$ . Then  $S$  is called cover for (1.2.1), if

- (i)  $\sum_{j \in S} a_j > a_0$
- (ii)  $\sum_{i \in T_1} a_j \leq a_0, j \in S \text{ and } T_1 = S - \{j\}$

Let  $S' = \{j \in N - S \mid a_j \geq a_{j_1}\}$

Where  $j_1 = \max_{j \in S} a_j, E(S) = S \cup S', T_2 = N - E(S)$

- (iii) If  $T_2 \neq \Phi$ . Then  $\sum_{i \in T_3} a_i \leq a_0, T_3 = (S - \{j_1\}) \cup \{i_1\}$ , where  $i_1$

Defined by  $a_{i_1} = \max_{j \in T_2} a_j$

**Definition 2:** A strong cover S of (1) is called a strong q-cover of (1.2.1) if  $j_1=q$ .

Definition 1.2.3. Given a strong q-cover S, the set E(S) is called the extension of S, where  $S' = \{j \in N - S \mid a_j \geq a_{j_1}\}$

Let  $\delta$  be the set of all strong cover of (1.2.1). Balas and Jeroslow [5] have shown that if  $\delta$  be the family of minimal covers S for (1.2.1), then (1.2.1) is wquivalent to the set of (canonical) inequalities.

$$\sum_{j \in E(S)} x_j \leq |S| - 1, \text{ for all } S \in \delta \tag{2}$$

In the sense that  $x \in R^n$ ,  $x_j = 0$  or  $1$ ,  $j \in N$ , satisfies (1.2.1) if and only if it satisfies (1.2.2). Further it was shown that (1.2.1) is also equivalent to the set

$$\sum_{j \in S} x_j \leq |S| - 1, S \in \delta \tag{3}$$

It was empirically observed that strong canonically inequalities often defined facets of convex hull of 0 -1, points satisfying (1.2.1), i.e. facet of the Knapsack polytope.

$$P = \text{conv} \{x \in R^n \mid \sum_{j \in R^n} a_j x_j \leq a_0, x_j = 0 \text{ or } 1, j \in N\}$$

### 3. Canonical facets of the Knapsack Polytope

Let d be the dimension of P. The inequality  $\sum_{j \in M} x_j \leq K^k$  (4)

Where  $M \subseteq N$  and  $k \geq 0$ , is said to define a facet [(d-1) – dimensional face] of P if

and only if the half space defined by  $\sum_{j \in M} x_j = k$  (5)

Contains exactly  $d$  (affinely) independent points of  $P$ . Proposition 1. 3. 1.  $d = n - n'$ , where  $n' = |N'|$  and  $N' = \{j \in N \mid a_j > a_0\}$

Proof.  $d$  being the dimension of  $P$ ,  $d \geq n - n'$ , since  $P$  contains  $n - n'$  unit vectors  $e_j$ ,  $j \in N - N'$ . Also  $d \leq n - n'$ , since  $x_j = 0$  for all  $J \in n'$  for any  $x \in P$ .

Hence  $d = n - n'$

**Proposition 1:** The inequality  $x_j \leq 1$  defines facet of  $P$  if and only if

$$a_j^* a_j \leq a_0$$

Where  $a_j^* = \max_{i \in N - \{j\}} a_i$  (1.3.2)

Proof. Let  $e_i$  be the  $i$ th unit vector,  $i \in N$ . Let  $e_j$  be any arbitrary unit vector  $j \in N$ . The vectors  $e_j, e_i + e_j$  for all  $i \in N - \{j\}$  are also linearly independent which are contained in  $P$  for which the inequality (1.3.2) holds. Thus if (1.3.2) holds  $x_j \leq 1$  defines a facet of  $P$ .

Suppose  $x_j \leq 1$  defines a facet of  $P$ . Then  $x_j^i = 1$  for  $n$  linearly independent vectors  $x^i$  of  $P$  which are the vertices of  $P$  for  $i = 1, 2, \dots, n$ . Let  $X$  be the matrix whose rows are the vectors  $x^i$ . If (1.3.2) does not hold good, then  $x_j^i = 0, i = 1, 2, \dots, n$ , then the matrix  $X$  is a singular matrix. But  $X$  is not singular. So (1.3.2) must hold.

**Example-1:** Let  $P$  be the convex hull of 0-1 points satisfying

$$10x_1 + 10x_2 + 4x_3 + 7x_4 + 2x_5 + 6x_6 + x_7 + x_8 \leq 15.$$

Then  $x_j \leq 1$ , defines a facet of  $p$  for  $j = 3, 5, 7, 8$  but not for  $j = 1, 2, 4, 6$ . Following algorithm is being introduced to determine facets of the convex hull  $P$ .

**Algorithm 1:**

Step 1... Find  $j^* \in N$  such that

$$a_{j^*} = \max_{i \in N - \{j\}} a_i$$

Step 2: is  $a_{j^*} + a_j \leq a_0$ ?

If yes: then  $x_j \leq 1, j \in N$  is a facet.

If no: then  $x_j \leq 1, j \in N$  is not a facet.

Step 3: Redefine N:  $N = N - \{j\}$

Is  $N = \emptyset$ ?

If yes: Stop, all facets have been found out.

If no: Go to step 2.

**Theorem 2:** If  $|M| \geq k+1 \geq 2$  and (1.3.1) defines a facet of P, then for each  $i \in N$ , P has a vertex  $\bar{x}$  satisfying (1.3.1a) and such that  $\bar{x}_i = 1$ , and for each  $i \in M$ , P has a vertex  $x$  satisfying (1.3.1a) and such that  $x_i = 0$

Proof . We shall prove this theorem by the method of contradiction. Since (1.3.1) defines a facet of P, (1.3.1a) is satisfied by  $n$  linearly independent vertices  $x^h, h=1, \dots, n$  of P. If  $X$  is the  $n \times n$  matrix whose rows are these vertices  $x^h$ , then  $x_i^h = 0, h=1, 2, \dots, n$  for some  $i \in N$  implies that  $x$  is singular. Hence for each  $i \in N, x_i^h = 1$  for some  $h \in \{1, \dots, n\}$ .

Also, if  $X_M$  is the sub matrix of  $X$  whose columns are indexed by  $M$ , then each row of  $X_M$  has exactly  $k$  entries equal to 1, and therefore,  $x_i^h = 1$  for  $h = 1, \dots, n$  and some  $i \in M$ , then the  $i$ th column of  $X_M$  is the sum of the remaining columns of  $X_M$  divided by  $k-1$ , hence the columns of  $X$  are linearly dependent. Thus for each  $i \in M, x_i^h = 0$  for some  $h \in \{1, \dots, n\}$ .

The following algorithm, for facets is given according above theorem.

**Algorithm 2**

Step 1: Choose  $S \subset N$  a strong cover for Knapsack polytope  $P$ .

Step 2: Find out  $K^k$  from  $|S| = k+1$

Choose  $j_1$  such that  $a_{j_1} = \max_{j \in S} a_j$

Set  $M = S \cup \{j \in N - S \mid a_j \geq a_{j_1}\}$

Is  $M \geq 2$ ?

If yes: Go to step 3.

If no equation (1.3.1) does not defines a facet,

go to step 4.

Step 3: Set  $T^1 = \{S - \{j_1\}\}$

Choose  $j_2$  such that  $a_{j_2} = \max_{j \in T^1} \{a_j\}$

Set  $T^2 = \{S - \{j_1, j_2\}\} \cup \{1\}$

Is  $\sum_{j \in T^2} a_j \leq a_0$

If yes: Infer equation (1.3.1) defines a facet.

If no: Equation (1.3.1) does not define a facet, go to step 4.

Step 4: Redefine  $S$ :  $N - S$

Is  $S = \Phi$

If yes: Stop, All facets has been found out.

If no: Go to Step 2.

**Example 2:** Let P be the convex hull of 0-1 points satisfying

$$4x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 + x_6 \leq 5.$$

Here,  $N = \{1, 2, 3, 4, 5, 6\}$ ,  $a_0=5$ ,  $a_1=4$ ,  $a_2=4$ ,  $a_3=3$ ,  $a_4=2$ ,  $a_5=2$ ,  $a_6=1$

Let us choose  $S = \{3, 4\}$

$$\therefore |S| = 2 \Rightarrow k+1=2 \Rightarrow k=1.$$

$$a_{j_1} = \max_{j \in S} a_j \Rightarrow a_{j_1} = a_3 \Rightarrow j_1 = 3$$

$$T^1 = S - \{j_1\} = \{4\}$$

$$a_{j_2} = \max_{j \in T^1} a_j \Rightarrow 3 = a_4 \Rightarrow j_2 = 4$$

$$S' = \{j \in N - S : a_j \geq a_j\}$$

$$= \{j \in \{1, 2, 5, 6\} : a_j \geq a_{j_1}\} = \{1, 2\}$$

$$M = E(S) = S \cup S' = \{3, 4\} \cup \{1, 2\} = \{1, 2, 3, 4\}$$

$$\therefore T^2 = (S - \{j_1, j_2\}) \cup \{1\} = (\{3, 4\} - \{3, 4\}) \cup \{1\} = \{1\}$$

$$\sum_{j \in T^2} a_j = a_1 = 4 < a_0 (= 5)$$

Thus the condition of Algorithm 1.3.1 is satisfied.

So for  $|M| \geq 2$ ,  $\sum_{j \in M} x_j \leq k$  is a facet

$x_1 + x_2 + x_3 + x_4 \leq 1$  is a facet

Let  $S = \{1, 2, 6\}$

$$\therefore |S| = 3 \Rightarrow k + 1 = 3 \Rightarrow k = 2$$

$$a_{j_1} = \max_{j \in S} a_j \Rightarrow a_{j_1} = 1 \Rightarrow j_1 = 1$$

$$T^1 = S - \{j_1\} = (2, 6)$$

$$a_{j_2} = \max_{j \in T^1} a_j \Rightarrow \max_{j \in \{2, \sigma\}} a_j = a_2 \Rightarrow j_2 = 2$$

$$S' = \{j \in N - S : a_j \geq a_{j_1}\} = \Phi$$

$$M = E(S) = S \cup S' = \{1, 2, 6\} \cup \Phi = \{1, 2, 6\}$$

$$\therefore T^2 = (S - \{j_1, j_2\}) \cup \{1\} = (\{1, 2, 6\} - \{1, 2\}) \cup \{1\} = \{1, 6\}$$

$$\sum_{j \in T^2} a_j = a_1 + a_6 = 4 + 1 = 5 = a_0 (= 5)$$

Thus Algorithm (1.3.2) is satisfied.

Here  $|M| = 3 > 2$

So  $\sum_{j \in M} x_j \leq k$  defines a facet of given inequality.

$x_1 + x_2 + x_6 \leq 2$  is facet.

Similarly by choosing  $\{1, 2, 6\}, \{2, 3, 6\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\}$  and  $\{4, 5, 6\}$  as strong cover we can develop the following facets by theorem (2.3.1)

$$x_1 + x_2 + x_6 \leq 2$$

$$x_1 + x_2 + x_3 + x_6 \leq 2$$

$$x_1 + x_2 + x_4 + x_6 \leq 2$$

$$x_1 + x_2 + x_5 + x_6 \leq 2$$



$$x_1 + x_2 + x_3 + x_4 + x_6 \leq 2$$

$$x_1 + x_2 + x_3 + x_5 + x_6 \leq 2$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 2$$

**Theorem 3:** The inequality

$$\sum_{j \in N} \pi_j x_j \leq \pi_0 \tag{1.3.6}$$

Where  $\pi_0 \geq 0$  is an integer, satisfied by all  $x \in P$ , if  $N$  can be partitioned into  $(q+1)$  subsets  $N_h, h = 0, 1, 2, \dots, q, 1 \leq q \leq \pi_0$  such that

$$(\alpha) \pi_j = h \text{ for all } j \in N_h, h = 0, 1, 2, \dots, q,$$

( $\beta$ )  $M = \bigcup_{h=1}^q N_h$  is the extension of some minimal cover for  $S$  for (1.2.1) such that

$$S \subseteq N \text{ and } |S| = \pi_0 + 1$$

$$(\gamma) N_0 = N - M, N_1 = M - Z, \text{ where } Z = \bigcup_{h=2}^q N_h$$

$$N_h = \{i \in N : \sum_{j \in S_{h+1}} a_j \leq a_i \leq \sum_{j \in S_h} a_j\}, h = 2, \dots, q$$

Where  $S_h$  is the set of the first  $h$  elements of  $S, h = 2, \dots, q+1$ . If in addition to

$$(\alpha), (\beta) \text{ and } (\gamma), \text{ one also has } (\eta) \sum_{j \in S - S_{h+1}} a_j + a_i \leq a_0 \text{ for all } i \in N_h = 0, 1, \dots, q$$

Then (1.3.6) defines a facet of  $P$ .

Theorem (1.3.2) lays the ground work for generating a family  $T$  of valid cutting planes. i.e. inequalities satisfied by all 0-1 points satisfying (1.2.1), most of which are facets of  $P$ . the procedure is given by the following algorithms as the proof of the theorem is described in Balas [2]

**Algorithm 3**

Step 1: Select  $S \subseteq N$

Is  $\sum_{j \in S} a_j > a_0$  ?

If yes, Take S as a cover

If no: Stop, infer S is not a cover. Go to step 4.

Step 2 set  $T_1 = S - \{j\}, j \in S$

Is  $\sum_{i \in T} a_j \leq a_0$

If yes: Take S as minimal cover.

If no: stop, infer S is not minimal cover.

Go to step 4.

Step3 (Choose  $j_1$  such that  $a_{j_1} = \max_{j \in S} a_j$ )

Set  $E(S) = S \cup \{j \in N - S \mid a_j \geq a_{j_1}\}$

Set  $T_2 = N - E(S)$

Is  $T_2 \neq \Phi$

If yes: Go to step 4

If no: Infer S is not a strong cover.

Step 4: Choose  $i_1$ , such that,  $a_{i_1} = \max_{j \in T^2} a_j$

Set  $T_3 = (S - \{j_1\}) \cup \{i_1\}$

Is  $\sum_{j \in I_3} a_j \leq a_0$

If yes: Take S as a strong cover.

If no: S is not strong cover.

Step5. Redefine S: N-S

S: S-{j} ∪ {i | i ∈ N}

Step 6: Is S ≠ Φ ?

If yes: Stop, all strong covers are found

If no: Go to step 1.

**Algorithm 4:**

Step1: Set  $\pi_0 = |S| - 1$ ,  $S \in T =$  family of cuts or strong covers

$\pi_j = h, j \in N_h, h = 0, 1, \dots, q$

$N_0 = N - E(S), N_1 = E(S) - \cup_{h=2}^q N_h$

$N_h = \{i \in N \mid \sum_{j \in S_h} a_j \leq a_i \leq \sum_{j \in S_{h+1}} a_j\}, h=2, \dots, q$

$S_h$  is the set of first h elements of S for  $h = 2, \dots, q+1$

Step2. is  $\sum_{j \in S - S_{h+1}} a_j + a_i \leq a_0$

For  $I \in N_h, h = 1, 2, \dots, q$ .

If yes:  $\sum_{j \in N_h} \pi_j x_j \leq \pi_0$  is a valid cut

If no: S is not a facet.

Step3. Redefine  $S = F - S$

Is  $S = \Phi$

If yes: Stop, all facets are found.

If not: Go to step 1.

**Example 3:** Let P be the convex hull of 0-1 points satisfying.

$$10x_1 + 8x_2 + 6x_3 + 5x_4 + 3x_5 + 3x_6 + 3x_7 + 2x_8 + 2x_9 + x_{10} \leq 11$$

Find out the facets along with strong covers of above inequality.

Answer: According to given inequality, we have  $a_1 = 10, a_2 = 8, a_3 = 6$

$a_4 = 5, a_5 = 3, a_6 = 3, a_7 = 3, a_8 = 2, a_9 = 2, a_{10} = 2$  and  $a_0 = 11$

So,  $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Table 1.3.1: Lists the family of cutting planes characterized by theorem 1.3.2, along with the strong cover from which they are generated of the 24 members of the family all are facets of P. The table is to be read as follows.

Take line 1:  $S = \{2, 3\}$  is a cover, since  $a_2 + a_3 = 8 + 6 > a_0 = 11$ ; a minimal cover since  $6 \leq 11$  and  $8 \leq 11$ : a strong cover as  $a_3 + a_4 = 6 + 4 \leq 10$ . Extension of S is  $E(S) = S \cup S' = \{2, 3\} \cup \{1\} = \{1, 2, 3\}$  and  $N_0 = N - E(S) = \{4, 5, 6, 7, 8, 9, 10\}$ . Since S contains only two elements, so it is impossible to set  $S_3 \Rightarrow N_h = 0$  for  $h > 1$ .

$$\therefore N_1 = E(S) - \bigcup_{h=2}^q N_h + E(S) - 0 = E(S) = \{1, 2, 3\}$$

Hence the first cut in the family F is

$$x_1 + x_2 + x_3 \leq 1$$

Which defines a facet of P, since  $S-S_2 = \Phi$  and  $a_1 = 10 \leq a_0 = 11$ .

Now let us consider a separate set as  $\{4, 5, 6, 7\}$  from below. It is a cover since

$a_4 + a_5 + a_6 + a_7 = 4 + 3 + 3 + 2 = 12 > 10$ . Further,  $E(S) = \{1, 2, 3, 4, 5, 6, 7\}$  and  $N_0 = N - E(S) = \{8, 9\}$ ,  $N_2 = \{1, 2\}$ , since  $a_4 + a_5 \leq a_i \leq a_4 + a_5 + a_6$  for  $i = 1, 2$ .  $N_3 = \Phi$ , since

$$a_4 + a_5 + a_6 < a_i \leq a_4 + a_5 + a_6 + a_7 \text{ for } i=1.$$

$$N_h = 0, h > 2: N_1 = E(S) - N_2 = \{3, 4, 5, 6, 7\}$$

$$\pi_j = h, j \in N_h, h = 0, 1, 2, \pi_0 = |S| - 1 = 4 - 1 = 3$$

$$j \in N_1 = \{3, 4, 5, 6, 7\}$$

$$\pi_3 = \pi_4 = \pi_5 = \pi_6 = \pi_7 = h = 1.$$

$$j \in N_2 = \{1, 2\}$$

$$\pi_2 = \pi_1 = h = 2.$$

So the cut associated with S is

$$2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 3.$$

#### 4. Conclusion

In this paper we have attempted to characterize the convex hull of 0-1 solutions to linear inequalities to a linear inequality. We are motivated by the work of Balas [4] and Wolsy [80]. The purpose of our research was to present algorithms for suitable computer programs for enumerating strong cover planes directly from the definitions

Table 1- The results from example 3

Strong cover{S}	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$	$\pi_7$	$\pi_8$	$\pi_9$	$\pi_{10}$	$\pi_0$
1,5	1	0	0	0	1	0	0	0	0	0	1
1,6	1	0	0	0	0	1	0	0	0	0	1
1,7	1	0	0	0	0	0	1	0	0	0	1
1,8	1	0	0	0	0	0	0	1	0	0	1
1,9	1	0	0	0	0	0	0	0	1	0	1
2,3	1	1	1	0	0	0	0	0	0	0	1
2,4	1	1	0	1	0	0	0	0	0	0	1
1,8,10	1	0	0	0	0	0	0	1	0	1	2
1,9,10	1	0	0	0	0	0	0	0	1	1	2
2,5,8	1	1	0	0	1	0	0	1	0	0	2
2,5,6	1	1	0	0	1	0	0	0	1	0	2
2,5,10	1	1	0	0	1	0	0	0	0	1	2
3,4,5	1	1	1	1	1	0	0	0	0	0	2
3,4,6	1	1	1	1	0	1	0	0	0	0	2
3,4,7	1	1	1	1	0	0	1	0	0	0	2
3,5,6	2	1	1	0	0	1	0	0	0	0	2
3,5,7	2	1	1	0	1	0	1	0	0	0	2
3,5,8	2	1	1	0	1	0	0	1	0	0	2

4, 5, 6, 7	2 2 1 1 1 1 1 0 0 0 3
4, 5, 6, 8	2 2 1 1 1 1 0 1 0 0 3
4, 5, 6, 9	2 2 1 1 1 1 0 0 1 0 3
4, 5, 6, 10	2 2 1 1 1 1 0 0 0 1 3
4, 6, 7, 8	2 2 1 1 0 1 1 1 0 0 3
4, 6, 7, 9	2 2 1 1 0 1 1 0 1 0 3
4, 6, 7, 10	2 2 1 1 0 1 1 0 0 1 3
5, 6, 7, 8, 9	3 2 2 1 1 1 1 1 1 0 4

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