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The Full Averaging of Fuzzy Differential Inclusions

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Abstract

In this paper the substantiation of the method of full averaging for fuzzy differential inclusions is considered. These results generalize the results of [17,20] for differential inclusions with Hukuhara derivative and of [18] for fuzzy differential equations.

Keywords: fuzzy differential inclusions; averaging.

1. Introduction

When a real world problem is transferred into a deterministic initial value problem of ordinary differential equations (ODE), namely

$$x' = f(t, x), x(t_0) = x_0,$$

we cannot usually be sure that the model is perfect. If the underlying structure of the model depends upon subjective choices, one way to incorporate these into the model, is to utilize the aspect of fuzziness [22,28], which leads to the consideration of fuzzy differential equations (FDE). The intricacies involved in

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incorporating fuzziness into the theory of ODE pose a certain disadvantage and other possibilities are being explored to address this problem. One of the approaches is to connect FDE to multivalued differential equations and examine the interconnection between them [7,8,9,10,15,16,18,21,23,26,27]. The other approach is to transform FDE into differential inclusion with the fuzzy right-hand sides so as to employ the existing theory of differential inclusions [1,3,4,6,12,13,14].

In [24] the concept of fuzzy differential inclusion is entered, theorems of existence and continuous dependence on parameter of classical solutions of fuzzy differential inclusions are received. In [19,25] the concepts of ordinary, generalized and Kvazi solutions of fuzzy differential inclusions are entered, the relationship between sets of such solutions is investigated.

In this paper the substantiation of the method of full averaging for fuzzy differential inclusions is considered. These results generalize the results of [17,20] for differential inclusions with Hukuhara derivative and of [18] for fuzzy differential equations.

2. Main definitions

Let $conv(\mathrm{R}^n)$ be the family of all nonempty compact convex subsets of R^n with the Hausdorff metric

$$h(A,B)=\max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n.

Let E^n be the family of mappings $x:\mathrm{R}^n\to[0,1]$ satisfying the following conditions:

1) x is normal, i.e. there exists a $y_0\in\mathrm{R}^n$ such that $x(y_0)=1$;

2) x is fuzzy convex, i.e. $x(\lambda y+(1-\lambda)z)\geq\min\{x(y),x(z)\}$ whenever $y,z\in\mathrm{R}^n$ and $\lambda\in[0,1]$;

3) x is upper semicontinuous, i.e. for any $y_0\in\mathrm{R}^n$ and $\varepsilon>0$ exists $\delta(y_0,\varepsilon)>0$ such that $x(y)<x(y_0)+\varepsilon$ whenever $\|y-y_0\|<\delta, y\in\mathrm{R}^n$;

4) the closure of the set $\{y \in \mathrm{R}^n : x(y) > 0\}$ is compact.

Let $\hat{0}$ be the fuzzy mapping defined by $\hat{0}(y) = 0$ if $y \neq 0$ and $\hat{0}(0) = 1$.

Definition 2.1 The set $\{y \in \mathrm{R}^n : x(y) \geq \alpha\}$ is called the α – level ,$[x]^\alpha$, of the mapping $x \in \mathrm{E}^n$ for $0 < \alpha \leq 1$. The closure of the set $\{y \in \mathrm{R}^n : x(y) > 0\}$ is called the 0 - level ,$[x]^0$, of the mapping $x \in \mathrm{E}^n$.

Theorem 2.1. [15] If $x \in \mathrm{E}^n$ then

1) $[x]^\alpha \in conv(\mathrm{R}^n)$ for all $0 \leq \alpha \leq 1$;

2) $[x]^{\alpha_2} \subset [x]^{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

3) if $\{\alpha_k\} \subset [0,1]$ is a nondecreasing sequence converging to $\alpha > 0$, then $[x]^\alpha = \bigcap_{k \geq 1} [x]^{\alpha_k}$.

Conversely, if $\{A^\alpha : 0 \leq \alpha \leq 1\}$ is the family of subsets of R^n satisfying the conditions 1 - 3 then there exists $x \in \mathrm{E}^n$ such that $[x]^\alpha = A^\alpha$ for $0 < \alpha \leq 1$ and $[x]^0 = \overline{\bigcup_{0<\alpha\leq 1} A^\alpha} \subset A^0$.

Define the metric $D : \mathrm{E}^n \times \mathrm{E}^n \to \mathrm{R}_+$ by the equation $D(x, y) = \sup_{\alpha \in [0,1]} h([x]^\alpha, [y]^\alpha)$.

Let I be an interval in R.

Definition 2.2. A mapping $f : I \to \mathrm{E}^n$ is called continuous at point $t_0 \in I$ provided for any $\varepsilon > 0$ there exists $\delta > 0$ such that $D(f(t), f(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta, t \in I$. A mapping $f : I \to \mathrm{E}^n$ is called continuous on I if it is continuous at every point $t_0 \in I$.

Definition 2.3. [15] A mapping $f : I \to \mathrm{E}^n$ is called measurable on I if for any $\alpha \in [0,1]$ the multivalued mapping $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable.

Definition 2.4. [15] A mapping $f : I \to \mathrm{E}^n$ is called integrably bounded on I if there exists a Lebesgue integrable function $k(t)$ such that $\| x \| \leq k(t)$ for all $x \in f_0(t), t \in I$.

Definition 2.5. [15] An element $g \in \mathrm{E}^n$ is called an integral of $f : I \to \mathrm{E}^n$ over I if $[g]^\alpha = (A)\int_I f_\alpha(t)dt$ for any $\alpha \in (0,1]$, where $(A)\int_I f_\alpha(t)dt$ is the Aumann integral [2].

Theorem 2.2. [15] If a mapping $f : I \to \mathrm{E}^n$ is measurable and integrably bounded then f is integrable over I.

Definition 2.6. A mapping $f : I \to \mathrm{E}^n$ is called absolutely continuous on I if there exists an integrable map $g : I \to \mathrm{E}^n$ such that

$$f(t) = f(t_0) + \int_{t_0}^{t} g(s)ds, \text{ for every } t_0, t \in I.$$

Definition 2.7. [15] A mapping $f : I \to \mathrm{E}^n$ is called differentiable at point $t_0 \in I$ if for any $\alpha \in [0,1]$ the multivalued mapping $f_\alpha(t)$ is Hukuhara differentiable at point t_0 [5] and the family $\{D_H f_\alpha(t_0) : \alpha \in [0,1]\}$ defines a fuzzy number $f'(t_0) \in \mathrm{E}^n$ (which is called a fuzzy derivative of $f(t_0)$ at point t_0). A mapping $f : I \to \mathrm{E}^n$ is called differentiable on I if it is differentiable at every point $t_0 \in I$.

Theorem 2.3. If a mapping $f : I \to \mathrm{E}^n$ is differentiable almost everywhere on I and its fuzzy derivative $f' : I \to \mathrm{E}^n$ is integrable on I, then for every $t \in I$ we have

$$f(t) = f(t_0) + \int_{t_0}^{t} f'(s)ds, t_0 \in I.$$

Let $comp(\mathrm{E}^n)$ $(conv(\mathrm{E}^n))$ be the family of all subsets F of the space E^n such that the family of all $\alpha-$ level sets of the elements from F is the nonempty compact (and convex) element in $comp(\mathrm{R}^n)$ (that is the element of $cc(\mathrm{R}^n)$ $(cocc(\mathrm{R}^n))$ [11]) with metric

$$d(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} D(a,b), \sup_{b\in B}\inf_{a\in A} D(a,b)\}.$$

Define also the distance from an element $x \in \mathrm{E}^n$ to a set $A \in comp(\mathrm{E}^n)$:

$$dist(x, A) = \min_{a\in A} D(x,a).$$

Consider in $comp(\mathrm{E}^n)$ the usual algebraic operations:

– addition : $F + G = \{f + g : f \in F, g \in G\}$;

– multiplication by scalars λ: $\lambda F = \{g = \lambda f : f \in F\}$.

The following properties hold [24]:

1) if $F, G \in comp(\mathrm{E}^n)[conv(\mathrm{E}^n)]$ then $F + G \in comp(\mathrm{E}^n)[conv(\mathrm{E}^n)]$;

2) if $F \in comp(\mathrm{E}^n)[conv(\mathrm{E}^n)]$ then $\lambda F \in comp(\mathrm{E}^n)[conv(\mathrm{E}^n)]$;

3) $F + G = G + F$;

4) $F + (G + H) = (F + G) + H$;

5) there exists the null element $\{\hat{0}\}$: $F+\{\hat{0}\}=F$;

4) $\alpha(\beta F)=(\alpha\beta)F$;

5) $1\cdot F=F$;

6) $\alpha(F+G)=\alpha F+\beta G$;

7) if $\alpha\geq 0,\beta\geq 0$ and $F\in conv(\mathrm{E}^n)$ then $(\alpha+\beta)F=\alpha F+\beta F$; otherwise $(\alpha+\beta)F\subset\alpha F+\beta F$.

Definition 2.8. A mapping $F:I\to comp(\mathrm{E}^n)$ is called the fuzzy multivalued mapping.

Definition 2.9. A fuzzy multivalued mapping $F:I\to comp(\mathrm{E}^n)$ is called measurable on I if the set $\{t\in I:F(t)\bigcap G\neq\varnothing\}$ is measurable for every $G\in comp(\mathrm{E}^n)$.

Definition 2.10. A fuzzy multivalued mapping $F:I\to comp(\mathrm{E}^n)$ is called continuous at point $t_0\in I$ provided for any $\varepsilon>0$ there exists $\delta(t_0,\varepsilon)>0$ such that $d(F(t),F(t_0))<\varepsilon$ whenever $|t-t_0|<\delta, t\in I$. A fuzzy multivalued mapping $f:I\to\mathrm{E}^n$ is called continuous on I if it is continuous at every point $t_0\in I$.

Definition 2.11. A function $f:I\to\mathrm{E}^n$ is called a selector of a fuzzy multivalued mapping $F:I\to comp(\mathrm{E}^n)$ if $f(t)\in F(t)$ for almost every $t\in I$.

Obviously a selector $f(t)$ always exists as the set $F(t)$ is not empty for all $t\in I$.

Define the integral of $F:I\to comp(\mathrm{E}^n)$ over I:

$$\int_I F(t)dt=\left\{\int_I f(t)dt:\ f(t)\in F(t)\ \text{ almost everywhere on }\ I\right\}.$$

Consider the fuzzy differential inclusion

$$x'\in F(t,x),\quad x(t_0)=x_0, \tag{1}$$

where $t\in I$ is time, $x\in G\subset\mathrm{E}^n$ is a phase variable, the initial conditions $t_0\in I, x_0\in G$ and $F:I\times G\to comp(\mathrm{E}^n)$ is a fuzzy multivalued mapping.

Definition 2.12. An absolutely continuous mapping $x(t), x(t_0)=x_0$, is called an ordinary solution of the differential inclusion (1) if

1) $x(t)\in G$ for all $t\in I$;

2) $x'(t)\in F(t,x(t))$ almost everywhere on I.

Theorem 2.4. Let the fuzzy multivalued mapping $F:I\times G\to conv(\mathrm{E}^n)$ satisfy the following conditions:

1) $F(\cdot,x)$ is measurable on I;

2) $F(t,\cdot)$ satisfies the Lipschitz condition with the constant $k>0,$ i.e. for all $(t,x),(t,y)\in I\times G$ the following inequality holds

$$d(F(t,x),F(t,y))\le kD(x,y);$$

3) there exists an absolutely continuous mapping $y(t),\ y(t_0)=y_0,$ such that $D(y(t),x_0)\le b$ and $dist(y'(t),F(t,y(t)))\le \eta(t)$ for all $t:|t-t_0|\le a$, where the function $\eta(t)$ is Lebesgue summable.

Then there exists a solution $x(t)$ of the fuzzy differential inclusion (1) on the interval $[t_0,t_0+\sigma]$ such that $D(x(t),y(t))\le r(t),$ where

$$r(t)=r_0e^{k(t-t_0)}+\int_{t_0}^{t}e^{k(t-s)}\eta(s)ds, r_0=D(x_0,y_0),$$

$$\sigma=\min\left\{a,\frac{b}{M}\right\}, M=\max_{(t,x)\in I\times G}d(F(t,x),\{\hat{0}\}).$$

3. Main Results

Consider the fuzzy differential inclusion

$$x'\in \varepsilon F(t,x),\quad x(0)=x_0, \tag{2}$$

where $t\in \mathrm{R}_+$ is time, $x\in G\in comp(\mathrm{E}^n)$ is a phase variable, $\varepsilon>0$ is a small parameter and $F:\mathrm{R}_+\times\mathrm{E}^n\to conv(\mathrm{E}^n)$ is a fuzzy multivalued mapping.

Let us associate with the inclusion (2) the following averaged fuzzy differential inclusion

$$\xi'\in \varepsilon\overline{F}(\xi),\quad \xi(0)=x_0, \tag{3}$$

where

$$\overline{F}(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(t,x)dt. \tag{4}$$

Theorem 3.1. Let in the domain $Q=\{t\in \mathrm{R}_+, x\in G\in comp(\mathrm{E}^n\})$ the following hold:

1) the fuzzy multivalued mapping $F(t,x)$ is continuous, uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ, i.e.

$$|F(t,x)|=d(F(t,x),\{\hat{0}\})\le M,$$

$$d(F(t,x),F(t,y))\le \lambda D(x,y);$$

2) uniformly with respect to x in the domain G the limit (4) exists;

3) for any $x_0\in G'\subset G$ and $t\in \mathrm{R}_+$ the solutions of the inclusion (3)

together with a $\rho-$ neighborhood belong to the domain G.

Then for any $\eta \in (0,\rho]$ and $L>0$ there exists $\varepsilon^0(\eta, L)>0$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t\in[0,L\varepsilon^{-1}]$ the following statements fulfill:

1) for any solution $\xi(t)$ of the inclusion (3) there exists a solution $x(t)$ of the inclusion (2) such that

$$D(x(t),\xi(t)) \le \eta; \tag{5}$$

2) for any solution $x(t)$ of the inclusion (2) there exists a solution $\xi(t)$ of the inclusion (3) such that the inequality (5) holds.

Proof. From the conditions (1) and (2) it follows that the fuzzy multivalued mapping $\bar{F}(x)$ is uniformly bounded with constant M and satisfies the Lipschitz condition with constant λ. Really in view of the condition (2) of the theorem for any $\delta>0$ it is possible to find $T(\delta)>0$ such that for all $T>T(\delta)$ the estimate is fair:

$$d\left(\bar{F}(x),\frac{1}{T}\int_0^T F(t,x)dt\right)<\delta.$$

Then choosing $T>T(\delta)$ we obtain

$$\left|\bar{F}(x)\right| = d\left(\bar{F}(x),\{\hat{0}\}\right) \le d\left(\bar{F}(x),\frac{1}{T}\int_0^T F(t,x)dt\right)+d\left(\frac{1}{T}\int_0^T F(t,x)dt,\{\hat{0}\}\right)$$

$$<\delta+\frac{1}{T}\int_0^T d\left(F(t,x),\{\hat{0}\}\right)dt \le \delta+M;$$

$$d(\bar{F}(x'),\bar{F}(x'')) \le d\left(\bar{F}(x'),\frac{1}{T}\int_0^T F(t,x')dt\right)+d\left(\frac{1}{T}\int_0^T F(t,x')dt,\frac{1}{T}\int_0^T F(t,x'')dt\right)$$

$$+d\left(\frac{1}{T}\int_0^T F(t,x'')dt,\bar{F}(x'')\right)<2\delta+\frac{1}{T}\int_0^T d\left(F(t,x')dt,F(t,x'')\right)dt$$

$$\le 2\delta+\frac{1}{T}\int_0^T \lambda D(x',x'')dt = 2\delta+\lambda D(x',x'').$$

Since δ is chosen arbitrarily, in a limit we will receive:

$$\left|\bar{F}(x)\right| \le M, \quad d(\bar{F}(x'),\bar{F}(x'')) \le \lambda D(x',x'').$$

Let us prove the first statement of the theorem. Divide the interval $[0,L\varepsilon^{-1}]$ on the partial intervals with the points $t_i=\frac{Li}{m\varepsilon}, i=\overline{0,m};\ m\in \mathrm{N}$. Let $\xi(t)$ be a solution of the inclusion (3). Then there exists a measurable selector $v(t)$ of the fuzzy multivalued mapping $F(\xi(t))$ such that

$$\xi(t)=\xi(t_i)+\varepsilon\int_{t_i}^{t} v(s)ds, t\in[t_i,t_{i+1}], \xi(t_0)=x_0.$$

(6)

Consider the mapping

$$\xi^1(t)=\xi^1(t_i)+\varepsilon v_i(t-t_i),\ t\in[t_i,t_{i+1}], \xi^1(0)=x_0, \tag{7}$$

where

$$D\left(\frac{L}{m\varepsilon}v_i, \int_{t_i}^{t_{i+1}} v(s)ds\right)=\min_{v\in\overline{F}(\xi^1(t_i))}\left(\frac{L}{m\varepsilon}v, \int_{t_i}^{t_{i+1}} v(s)ds\right). \tag{8}$$

Denote by $\delta_i=D(\xi(t_i),\xi^1(t_i))$. As

$$D(\xi(t),\xi(t_i))=\varepsilon D\left(\int_{t_i}^{t} v(s)ds,\hat{0}\right)\le\varepsilon M(t-t_i)\le\frac{ML}{m}, \tag{9}$$

then

$$D(\xi(t),\xi^1(t_i))\le D(\xi(t),\xi(t_i))+D(\xi(t_i),\xi^1(t_i))\le\delta_i+\varepsilon M(t-t_i), t\in[t_i,t_{i+1}],$$

$$d(\overline{F}(\xi(t)),\overline{F}(\xi^1(t_i)))\le\lambda[\delta_i+\varepsilon M(t-t_i)]. \tag{10}$$

From (8) and (10) it follows that

$$\begin{aligned}D\left(\int_{t_i}^{t_{i+1}} v(s)ds,\frac{L}{\varepsilon m}v_i\right)&\le d\left(\int_{t_i}^{t_{i+1}}\overline{F}(\xi(t))dt,\int_{t_i}^{t_{i+1}}\overline{F}(\xi^1(t_i))dt\right)\\&\le\int_{t_i}^{t_{i+1}} d(\overline{F}(\xi(t)),\overline{F}(\xi^1(t_i)))dt\le\lambda\left[\delta_i(t_{i+1}-t_i)+\varepsilon M\,\frac{(t_{i+1}-t_i)^2}{2}\right]\\&=\lambda\left[\frac{L}{\varepsilon m}\delta_i+\frac{ML^2}{2\varepsilon m^2}\right].\end{aligned} \tag{11}$$

From (6), (7) and (11) we have

$$\begin{aligned}\delta_{i+1}&=D\left(\xi(t_{i+1}),\xi^1(t_{i+1})\right)=D\left(\xi(t_i)+\varepsilon\int_{t_i}^{t_{i+1}} v(s)ds,\xi^1(t_i)+\varepsilon v_i(t_{i+1}-t_i)\right)\\&\le\delta_i+\varepsilon\lambda\left[\frac{L}{\varepsilon m}\delta_i+\frac{ML^2}{2\varepsilon m^2}\right]=\frac{\lambda ML^2}{2m^2}+\left(1+\frac{\lambda L}{m}\right)\delta_i\\&\le\frac{\lambda ML^2}{2m^2}+\left(1+\frac{\lambda L}{m}\right)\left(\frac{\lambda ML^2}{2m^2}+\left(1+\frac{\lambda L}{m}\right)\delta_{i-1}\right)\\&\ \ \vdots\\&\le\left(1+\frac{\lambda L}{m}\right)^{i+1}\delta_0+\frac{\lambda ML^2}{2m^2}\sum_{k=0}^{i}\left(1+\frac{\lambda L}{m}\right)^k\end{aligned}$$

$$\leq \frac{ML}{2m}\left[\left(1+\frac{\lambda L}{m}\right)^{i+1}-1\right] \leq \frac{ML}{2m}(e^{\lambda L}-1), \quad i=\overline{0,m-1}. \tag{12}$$

As

$$D(\xi^1(t),\xi^1(t_i)) = \varepsilon D(v_i,\hat{0})(t-t_i) \leq \varepsilon M(t-t_i) \leq \frac{ML}{m}, \tag{13}$$

then using (9) and (12)

$$D(\xi(t),\xi^1(t)) \leq D(\xi(t),\xi(t_i)) + D(\xi(t_i),\xi^1(t_i)) + D(\xi^1(t_i),\xi^1(t)) \leq \frac{ML}{2m}(e^{\lambda L}+3). \tag{14}$$

From the condition (2) of the theorem it follows that for any $\eta_1 > 0$ and fixed m there exists $\varepsilon^1(\eta_1, m) > 0$ such that

$$d\left(\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}} F(s,\xi^1(t_i))ds, \bar{F}(\xi^1(t_i))\right) \leq \eta_1. \tag{15}$$

Hence, there exists $v^i(t) \in F(t,\xi^1(t_i)), i=\overline{0,m-1}$, such that

$$D\left(\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}} v^i(s)ds, v_i\right) \leq \eta_1. \tag{16}$$

Consider the family of mappings

$$x^1(t) = x^1(t_i) + \varepsilon\int_{t_i}^{t} v^i(s)ds, t \in [t_i,t_{i+1}], x^1(0) = x_0.$$

From (16), (17) and (6) it follows that

$$D(x^1(t_i),\xi^1(t_i)) \leq L\eta_1, i=\overline{1,m}. \tag{18}$$

As

$$D(x^1(t),x^1(t_i)) = \varepsilon D\left(\int_{t_i}^{t} v^i(s)ds,\hat{0}\right) \leq \varepsilon M(t-t_i) \leq \frac{ML}{m}, t \in [t_i,t_{i+1}],$$

then from (13) and (18) we have

$$D(x^1(t),\xi^1(t)) \leq \frac{2ML}{m} + L\eta_1 \tag{19}$$

and

$$d(F(t,x^1(t)),F(t,\xi^1(t_i))) \leq D(F(t,x^1(t)),F(t,x^1(t_i))) + D(F(t,x^1(t_i)),F(t,\xi^1(t_i))) \leq \lambda L\left(\frac{M}{m}+\eta_1\right). \tag{20}$$

Taking into consideration the choice of the mapping $v^i(t)$, (16) and (20), we have

$$dist(\dot{x}^1(t), \varepsilon F(t, x^1(t))) \le \varepsilon\lambda L\left(\frac{M}{m}+\eta_1\right). \tag{21}$$

According to the theorem 2.4 there exists such a solution $x(t)$ of the inclusion (2), that

$$D(x(t), x^1(t)) \le \varepsilon\lambda L\left(\frac{M}{m}+\eta_1\right)\int_0^t e^{\lambda\varepsilon(t-s)}ds \le L\left(\frac{M}{m}+\eta_1\right)(e^{\lambda L}-1). \tag{22}$$

From the estimates (14), (19) and (22) follows that

$$\begin{aligned} D(\xi(t), x(t)) &\le D(\xi(t), \xi^1(t)) + D(x(t), x^1(t)) + D(\xi^1(t), x^1(t)) \\ &\le \frac{ML}{2m}(e^{\lambda L}+3) + \frac{2ML}{m} + L\eta_1 + L\left(\frac{M}{m}+\eta_1\right)(e^{\lambda L}-1) \\ &= \frac{ML}{2m}(3e^{\lambda L}+5) + Le^{\lambda L}\eta_1. \end{aligned} \tag{23}$$

Choosing

$$m \ge \frac{ML}{\eta}(3e^{\lambda L}+5), \quad \eta_1 \le \frac{\eta}{2Le^{\lambda L}},$$

from (23) we get the first statement of the theorem.

The proof of the second part of the theorem is similar to the proof of the first one.

Remark 3.1. If the condition (3) doesn't hold it can be replaced by the following condition:

3') for any $x_0 \in G' \subset G$ the solutions of the inclusion (3) together with a $\rho-$ neighborhood belong to the domain G for $\tau \in [0, L^*],$ where $\tau = \varepsilon t.$

Then for any $\eta \in (0, \rho]$ and $L \in [0, L^*]$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the statements (1) and (2) of the theorem 3.1 fulfill.

In case when there is no uniform convergence in (4), the following theorem holds:

Theorem 3.3. Let in the domain Q the following hold:

1) the fuzzy multivalued mapping $F(t, x)$ is continuous, locally satisfies the Lipschitz condition in x;

2) in every point $x \in G$ the limit (4) exists;

3) for any $x_0 \in G' \subset G$ and $t \in \mathrm{R}_+$ the solutions of the inclusion (3)

together with a $\rho-$ neighborhood belong to the domain G.

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L, x_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the statements (1) and (2) of the theorem 3.1 fulfill.

Proof. Consider the set

$$G(L, x_0) = closure\{x \in \mathrm{E}^n : \text{there exists a solution } \xi(\tau), \tau = \varepsilon t \text{ of } (3) \text{ such that } D(x, \xi(\tau)) \le \rho, \tau \in [0, L]\}.$$

The set $G(L, x_0) \subset G$ is compact. Hence the limit (4) exists uniformly with respect to $x \in G(L, x_0)$. As at the proof of the theorem 3.1 it is enough to consider the domain $Q(L, x_0) = \{t \in \mathrm{R}_+, x \in G(L, x_0)\}$ the statements of the theorem 3.2 follow from the justice of the theorem 3.1 for the domain $Q(L, x_0)$.

Remark 3.2. The estimates received in the theorem 3.2 qualitatively differ from the corresponding estimates of the theorem 3.1. The external coincidence of the statements of theorems 3.1 and 3.2 leads to their wrong understanding. Really, the theorem 3.1 affirms that the inequality (?) holds uniformly for all solutions $x(t)$ and $\xi(t)$ with coincident initial conditions, i.e. the existence of $\varepsilon(\eta, L)$ is affirmed. The estimate received in the theorem 3.2 is fair only for solutions $x(t)$ and $\xi(t)$ beginning in the fixed initial point x_0, i.e. the existence of $\varepsilon(x_0, \eta, L)$ is affirmed.

If the fuzzy multivalued mapping $F(t, x)$ is periodic in t, one can receive the more exact estimate.

Theorem 3.3. Let in the domain Q the following hold:

1) the fuzzy multivalued mapping $F(t, x)$ is continuous, uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ;

2) the fuzzy multivalued mapping $F(t, x)$ is 2π - periodic in t;

3) for any $x_0 \in G' \subset G$ and $t \in \mathrm{R}_+$ the solutions of the inclusion (3) together with a $\rho-$ neighborhood belong to the domain G.

Then for any $L > 0$ there exist $\varepsilon^0(L) > 0$ and $C(L) > 0$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements fulfill:

1) for any solution $x(t)$ of the inclusion (2) there exists a solution $\xi(t)$ of

the inclusion (3) such that

$$D(x(t),\xi(t)) \le C\varepsilon; \tag{24}$$

2) for any solution $\xi(t)$ of the inclusion (3) there exists a solution $x(t)$ of the inclusion (2) such that the inequality (24) holds.

Proof. Note that in the case when the fuzzy multivalued mapping $F(t,x)$ is $2\pi-$ periodic in t we have

$$\bar{F}(x) = \frac{1}{2\pi}\int_0^{2\pi} F(t,x)dt.$$

Let us prove the first statement of the theorem. Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = 2\pi i,\ i = 0,1,\dots$. Let $x(t)$ be a solution of the inclusion (2). Then there exists a measurable selector $v(t)$ of the fuzzy multivalued mapping $F(t,x(t))$ such that

$$x(t) = x(t_i) + \varepsilon\int_{t_i}^{t} v(s)ds,\ t\in[t_i,t_{i+1}],\ x(0) = x_0. \tag{25}$$

Consider the mapping

$$x^1(t) = x^1(t_i) + \varepsilon\int_{t_i}^{t} v^1(s)ds,\ t\in[t_i,t_{i+1}],\ x^1(0) = x_0, \tag{26}$$

where $v^1(t)$ is the measurable selector of the multivalued mapping $F(t,x^1(t_i))$ such that

$$D(v(t),v^1(t)) = \min_{v^1\in F(t,x^1(t_i))} D(v(t),v^1). \tag{27}$$

Denote by $\delta_i = D(x(t_i),x^1(t_i))$, then we have

$$\begin{aligned} D(v(t),v^1(t)) &\le d(F(t,x(t)),F(t,x^1(t_i))) \le \lambda D(x(t),x^1(t_i)) \\ &\le \lambda\left[D(x(t),x(t_i)) + D(x(t_i),x^1(t_i))\right] \\ &\le \lambda\left[\varepsilon\int_{t_i}^{t} D(v(s),\hat{0})ds + \delta_i\right] \le \lambda[\delta_i + \varepsilon M(t-t_i)]. \end{aligned}$$

Therefore from (25) and (26) it follows

$$\begin{aligned} \delta_{i+1} &= D(x(t_{i+1}),x^1(t_{i+1})) = D\left(x(t_i) + \varepsilon\int_{t_i}^{t_{i+1}} v(s)ds,\ x^1(t_i) + \varepsilon\int_{t_i}^{t_{i+1}} v^1(s)ds\right) \\ &\le \delta_i + \varepsilon\int_{t_i}^{t_{i+1}} D(v(s),v^1(s))ds \le \delta_i + \varepsilon\lambda\int_{t_i}^{t_{i+1}} [\delta_i + \varepsilon M(s-t_i)]ds \\ &= \delta_i + 2\pi\varepsilon\lambda\delta_i + \varepsilon^2\lambda M\left.\frac{(s-t_i)^2}{2}\right|_{t_i}^{t_{i+1}} = \delta_i(1+2\pi\varepsilon\lambda) + 2\pi^2\varepsilon^2\lambda M. \end{aligned}$$

Hence, as $2\pi(i+1) \le L\varepsilon^{-1}$, we get

$$\begin{aligned}\delta_{i+1} &\le (1+2\pi\varepsilon\lambda)\delta_i + 2\pi^2\varepsilon^2\lambda M \\ &\le (1+2\varepsilon\lambda\pi)((1+2\pi\varepsilon\lambda)\delta_{i-1} + 2\pi^2\varepsilon^2\lambda M\ ) + 2\varepsilon^2\lambda M\pi^2 \\ &\quad\vdots \\ &\le (1+2\pi\varepsilon\lambda)^{i+1}\delta_0 + 2\pi^2\varepsilon^2\lambda M\sum_{k=0}^{i}(1+2\pi\varepsilon\lambda)^k \\ &= 2\pi^2\varepsilon^2\lambda M\sum_{k=0}^{i}(1+2\pi\varepsilon\lambda)^k \\ &= 2\varepsilon^2\lambda M\pi^2\frac{(1+2\varepsilon\lambda\pi)^{i+1}-1}{2\varepsilon\lambda\pi} \\ &= \varepsilon M\pi\left((1+2\varepsilon\lambda\pi)^{i+1}-1\right) \le \varepsilon M\pi(e^{\lambda L}-1),\end{aligned}$$

i.e.

$$\delta_i \le M\pi(e^{\lambda L}-1)\varepsilon, \quad i = 0,1,... \tag{28}$$

Taking into account that for $t \in [t_i, t_{i+1}]$ the following inequalities hold

$$D(x(t), x(t_i)) \le 2\pi M\varepsilon, \quad D(x^1(t), x^1(t_i)) \le 2\pi M\varepsilon, \tag{29}$$

using (28) we obtain

$$D(x(t), x^1(t)) = \pi M(e^{\lambda L}+3)\varepsilon. \tag{30}$$

Calculate the value of the mapping $x^1(t)$ in the points t_{i+1}:

$$x^1(t_{i+1}) = x^1(t_i) + \varepsilon\int_{t_i}^{t_{i+1}} v^1(s)ds = x^1(t_i) + 2\varepsilon v_i\pi, \tag{31}$$

where $v_i \in \frac{1}{2\pi}\int_0^{2\pi} F(s, x^1(t_i))ds = \overline{F}(x^1(t_i)).$

Consider the mapping

$$\xi^1(t) = \xi^1(t_i) + \varepsilon v_i(t - t_i), \quad t \in [t_i, t_{i+1}], \quad \xi^1(0) = x_0. \tag{32}$$

It is obvious that $x^1(t_i) = \xi^1(t_i), \quad i = 0,1,\ldots$

From (29) - (32) we have

$$D(x^1(t), \xi^1(t)) \le 4\pi M\varepsilon. \tag{33}$$

As for $t \in [t_i, t_{i+1}], \quad i = 0,1,...$

$$D(\xi^1(t), \xi^1(t_i)) \le 2\pi M\varepsilon, d(\overline{F}(\xi^1(t_i)), \overline{F}(\xi^1(t))) \le 2\lambda\pi M\varepsilon,$$

then

$$dist\left(\xi^1(t), \varepsilon\overline{F}(\xi^1(t))\right) \le d\left(\varepsilon\overline{F}(\xi^1(t_i)), \varepsilon\overline{F}(\xi^1(t))\right) \le 2\lambda\pi M\varepsilon^2. \tag{34}$$

According to the theorem 2.4 from the inequality (34) follows that there exists such a solution $\xi(t)$ of the inclusion (3), such that

$$D(\xi(t),\xi^1(t)) \le 2\varepsilon^2\pi\lambda M\int_0^t e^{\lambda\varepsilon(t-s)}ds \le 2\varepsilon\pi M(e^{\lambda L}-1).$$

(35)

From (30), (33) and (35) follows that

$$D(x(t),\xi(t)) \le D(x(t),x^1(t)) + D(x^1(t),\xi^1(t)) + D(\xi^1(t),\xi(t))$$
$$\le \pi M\varepsilon\left(e^{\lambda L}+3\right) + 4\pi M\varepsilon + 2\varepsilon\pi M\left(e^{\lambda L}-1\right) = \pi M\varepsilon\left(3e^{\lambda L}+5\right).$$

Denote by $C_1 = \pi M\left(3e^{\lambda L}+5\right)$, then

$$D(x(t),\xi(t)) \le C_1\varepsilon. \tag{36}$$

The first part of the theorem is proved.

Taking any solution $\xi(t)$ of the inclusion (3) and making the calculations as before, it is possible to find a solution $x(t)$ of the inclusion (2) such that inequality similar to (36) with some constant C_2 is fair. Choosing $C = \max(C_1, C_2)$ we will receive the justice of all statements of the theorem.

4. Conclusion

It is also possible to use the partial averaging of the fuzzy differential inclusions, i.e. to average only some summands or factors. Such variant of the averaging method also leads to the simplification of the initial inclusion and happens to be useful when the average of some functions does not exist or their presence in the system does not complicate its research.

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