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A Review on sequencing approaches for mixed-model just-in-time production system

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Abstract

Research interests have been focused on the concept of penalizing jobs both for being early and for being tardy because not only of modern competitive industrial challenges of providing a variety of products at a very low cost by smoothing productions but also of its increasing and exciting computer applications. Here, sequencing approaches of the mixed-model just-in-time production systems is reviewed. In this note, realizing a need of critical review, a survey on the elegant mathematical models, methods and complexity of the mixed-model just-in-time sequencing problem with an insight into the existing analytical literature is given. The established research results together with open problems and possible extensions are presented.

Keywords*:* Mixed-model; just-in-time; non-linear integer programming.

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1. Introduction

Mixed-model assembly lines with negligible change-over costs between the products allow manufacturing of different products of a common base product in evenly distributed sequences on the same line [12]. Just-in-time production system, which requires producing only the necessary products in the necessary quantities at the necessary times often uses mixed-model assembly lines [47]. The problem of finding a sequence of different products distributed as evenly as possible is called the mixed-model just-in-time sequencing problem (MMJITSP).

This problem minimizes both the earliness and the tardiness penalties that respond to the customer demands for a variety of models without holding large inventories or incurring large shortages. This requires the production of each model in diversified small-lot instead of large-lots in a flow line. The MMJITSP has goals of keeping the rate of usage of parts as constant as possible and of smoothing the work overload on each workstation on the line [51]. The second goal was taken up in other two alternative sequencing approaches, mixed- model sequencing and car sequencing also. The mixed-model sequencing problem that considers other operational characteristics though limited to a small subset of them of the line also is to minimize sequence dependent work overload [6,12]. The car sequencing problem is to find a sequence of product copies to meet the demand for each copy of the product without violating the given rules for production options. The problem avoids work overload implicitly through the control of the work intensive product options [54].

This paper mainly focuses on the first goal of the problem. MMJIT system consists of a hierarchy of finite and distinct levels such as products, sub-assemblies, component parts, raw materials, etc. The sequence at the final level is crucial and affects the entire supply chain as all other levels are also inherently fixed because of the pull nature of the system. Minimization of the variation in demand rates for outputs of supplying processes is the output rate variation problem (ORVP) [34]. This is a multi-level problem. Minimization of the variation in the rate at which different products are produced on the line is the product rate variation problem (PRVP), a single-level problem [34]. Assumptions that the products require approximately the same number and mix of parts or that the parts of an output of a level other than the product level are dedicated to be assembled into a particular product (pegging) reduces ORVP into the PRVP [57]. Even special cases of the ORVP are computationally more challenging than the PRVP [34, 41]. The problem has been formulated as a non-linear integer programming with the objective of minimizing the deviation between the actual and the ideal production under the assumption that the system has sufficient capacity with negligible switch-over costs from one product to another and each product is produced in a unit time [47, 49, 48]. See also [32]. The solutions to this problem have been referred as level, balanced or fair sequences.

Since the problem has been dealt with in a great number of papers with heuristics and pseudo-polynomial exact solution procedures, it would be worth to have a synthesis of them. A number of survey papers has been appeared [34, 21, 65, 12]. A recent survey performs a systematic record of the academic efforts pertaining to the problem [12]. Moreover, another one reviews the problem to help bridge the gap between the academic literature and industry practice [65]. This survey covers almost complete works of the problem including unresolved cases focused on mathematically interesting base model of theoretical value together with real world applications.

The plan of the paper is as follows. Section 2 reviews the mathematical model. In Section 3, sequencing procedures have been studied. Sections $2-3$ study in detail the level scheduling problem with a goal of uniform usages of all parts. Section 4 relates the MMJITSP to the apportionment problem. Sections 5 and 6 summarize production smoothing with arbitrary non-zero processing time and setup time, and smoothing the work overload, respectively. The last section concludes the paper.

2. Mathematical Model Formulation

The system consists of L different production levels l, $l = 1, \dots, L$ with product level 1. Let d_{il} be the demand for part i of level l, $i = 1, \dots, n_l$, n_l the number of different parts of level l. By t_{ilp}, we represent the number of total units of part i at level l required to produce one unit of product p, $p = 1, \dots, n_1$ and then $d_{il} = \sum_{p=1}^{n_1} t_{ilp}\, d_{p1}$, the dependent demand for part i of level l determined by d_{p1}. Clearly, $t_{i1p} = 1$ for $i = p$, and 0 otherwise. Let $D_l = \sum_{i=1}^{n_l} d_{il}$ be total part demands of level l with demand ratio $r_{il} = \frac{d_{il}}{D_l}$ and then $\sum_{i=1}^{n_l} r_{il} = 1$ for $l = 1, \dots, L$. The time horizon in the product level is partitioned into D_1 units and there will be k complete units of various products p at level 1 during the first k units. This introduces the concept of a stage. The pull nature of the system implies that the lower level parts are pulled forward according to the need of the product level.

Let x_{ilk} be the quantity of part i produced at level l in the time units 1 through k and $y_{lk} = \sum_{i=1}^{n_l} x_{ilk}$ be the total quantity produced at level l during these time units. At level 1, $y_{1k} = \sum_{i=1}^{n_1} x_{ilk} = k$. The required cumulative production for part i of level $l \geq 2$ through k time units will be $x_{ilk} = \sum_{p=1}^{n_1} t_{ilp} x_{p1k}$. Consider f_i unimodal symmetric convex non-negative function with minimum 0 at 0, $i = 1, \dots, n_l$. Then the mathematical model for the ORVP [49, 39], is

$$\text{minimize } [\tilde{F} = \max_{i,l,k} f_i(x_{ilk} - y_{lk} r_{il})] \tag{2.1}$$

$$\text{minimize } [\tilde{G} = \sum_{k=1}^{D_1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} f_i(x_{ilk} - y_{lk} r_{il})] \tag{2.2}$$

$$\text{subject to} \quad x_{ilk} = \sum_{p=1}^{n_1} t_{ilp} x_{p1k}, \qquad i = 1, \dots, n_l; \quad l = 1, \dots, L; k = 1, \dots, D_1 \tag{2.3}$$

$$y_{lk} = \sum_{i=1}^{n_l} x_{ilk}, \qquad l = 2, \dots, L; \; k = 1, \dots, D_1 \qquad (2.4)$$

$$y_{1k} = \sum_{p=1}^{n_1} x_{p1k} = k, \quad k = 1, \dots, D_1 \qquad (2.5)$$

$$x_{p1k} \geq x_{p1(k-1)}, \qquad p = 1, \dots, n_1; \; k = 1, \dots, D_1 \qquad (2.6)$$

$$x_{p1D_1} = d_{p1}, x_{p10} = 0, \; p = 1, \dots, n_1 \qquad (2.7)$$

$$x_{ilk} \geq 0, \text{ integer}, , \qquad i = 1, \dots, n_l; \; l = 1, \dots, L; \; k = 1, \dots, D_1 \qquad (2.8)$$

Constraint (2.3) ensures that the necessary cumulative production of part i of level l by the end of time unit k is determined explicitly by the quantity of products produced at level 1. Constraints (2.4) and (2.5) show the total cumulative production of level l and level 1, respectively, during the time units 1 through k. Constraint (2.6) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (2.7) guarantees that the demands for each product are met exactly. Constraints (2.5), (2.6), (2.8) ensure that exactly one unit of a product is scheduled during one time unit in the product level.

Use of weights in the model is an essential feature. Weighted case of the problem is formulated with appropriate weights w_{il}. The selection of weights will be based on the total production at various levels, the relative importance of having good schedules at the various levels and the numerical values assigned to the weights [49, 41]. Weights can be used to smooth the variability and to prevent lower-level parts to be dominant over higher level parts in the measures at different levels. Use of weights to a part shows the relative importance of the part that will affect the sequencing of the product into which that part is to be assembled [57].

The mathematical model for the ORVP reduces to the mathematical model for the PRVP when only the product level is considered and the superfluous subscript 1 is dropped out.

This model minimizes the perennial objective functions, the bottleneck measure of deviation F that produces smooth sequence in every time unit and the total measure of deviations G that produces smooth sequence on the average [55]. In particular,

$$F_a = \max_{i,k} |x_{ik} - kr_i| \text{ and } F_s = \max_{i,k} (x_{ik} - kr_i)^2$$

$G_a = \sum_{k=1}^{D} \sum_{i=1}^{n} |x_{ik} - kr_i|$ and $G_s = \sum_{k=1}^{D} \sum_{i=1}^{n} (x_{ik} - kr_i)^2$

in PRVP denote some particular objectives. We denote suffix a for the absolute deviation objective and suffix s for the square deviation, for example, problem F_a for the problem PRVP with the objective function F_a and the constraints. With an appropriate weight w_i, $i = 1, \dots, n$, the weighted problem is formulated.

An alternative objective for the minimization of the deviation between the times at which a unit of a product be actually produced and the time at which the unit of the product is needed to be produced is intuitively similar to the PRVP [28].

3. Sequencing Procedures

3.1. Heuristic approach

The ORVP is computationally more challenging. The problem $\widetilde{G_s}$ is NP-hard in the ordinary sense as the NP-hard scheduling problem, around the shortest job reduces to the ORVP [34] and the problem Fea with only two levels is NP-hard in the strong sense as the strongly NP-hard 3-partition problem transforms into the ORVP in pseudo-polynomial time [41]. However, a number of heuristics gives rise to suboptimal solutions.

The goal chasing methods GCM I and GCM II used in Toyota [51], see also [32], construct a sequence filling one position at a time from first slot to the last one. The variability is considered at the sub-assembly level whereas the variability at the product level is ignored. GCM II compared to GCM I represents a decrease in computational time because the sum is formed only on the components of a given product in GCM II [59]. GCM I and GCM II are myopic. A myopic polynomial heuristic, extended goal chasing method (EGCM) that considers more levels, adopts GCM I and GCM II as a special case [49]. The myopia lies in the fact that it only takes one step. Taking two steps into account, the myopia can be reduced [9].

Three algorithms and two heuristics are formulated in [47]. The algorithm 1 and the algorithm 3 with heuristic 1 (MA3H1) consider the product rates, not the parts usage rates. It is a one-stage myopic heuristic with complexity O(nD). The algorithms may not yield feasible sequence but if feasible it is optimal, too. The algorithm 3 with heuristic 2 (MA3H2) is the improved two-stage heuristic with complexity O(n2D). MA3H2 is of highest quality for feasible solutions among GCM I, GCM II, MA3H1 and MA3H2 [59, 25].

Time spread (TS) heuristic employs similar procedure as GCM1 with function in which time required to assemble products are applied. Comparison of different methods through simulation analysis show that TS and MA3H2 seem to be effective [60].

Inman and Bulfin's earliest due date (EDD) rule based on ideal time of production of each product [28], Ding and Cheng's two-stage algorithm that minimizes the variation of the two stages [22, 23] and MA3H2 heuristic obtain good solutions. Modified forms of these, with appropriate weights, are useful alternatives for frequent updates of sequencing [15].

A local search heuristic that attempts to swap the order of assembly of a pair of products provides near-optimal sequence for realistic-size problems in a reasonable time. It may be extended considering release date and due date constraints [27].

The problem with a bicriterion objective of part usage and setup time has inversely correlated objective values. An efficient frontier, where simultaneously maximization of feasibility and minimization of setup is desired, is exploited. Such frontier is explored using heuristics such as tabu search, simulated annealing, genetic algorithm, ant colony optimization approach, beam search heuristic, artificial neural network etc [44, 45, 16, 43].

Suboptimal solutions using heuristics, for example, tabu search and branch and bound to the problem with the objective for parts usage and work load [48, 60], and [24]; parts usage and line length [6]; parts usage and line stoppage, [69, 32] can be obtained.

3.2. Dynamic programming

Let the demand vector at level 1 be $d = (d_1, \dots, d_{n_1})$ and the states in a schedule be $X = (x_1, \dots, x_{n_1})$ with $|X| = \sum_{i=1}^{n_1} x_i$ where x_i is the cumulative production of product i, $x_i \leq d_i$. Let e_i be the unit vector with n_1 entries all of which are zero except for a single 1 in the i[th] row, and $\Gamma = w_{il}(t_{ilp} - r_{il} \sum_{n=1}^{n_1} t_{nlp})$ and $\Omega = w_{il}^{\frac{1}{2}}(t_{ilp} - r_{il} \sum_{n=1}^{n_l} t_{nlp})$ be the matrices of dimension $n \times n_1$, $n = \sum_{l=1}^{L} n_l$. Let $\|\Gamma X\|_1$ with the maximum norm $\|x\|_1 = \max_i |x_i|$, $i = 1, \dots, n_1$ and $(\|\Omega X\|_2)^2$ with the Euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^{n_1} {x_i}^2}$ be the maximum of absolute deviation and the sum of square deviations of actual production from the ideal production over all parts and products, respectively, where X is the amount of product produced. Define $\o(X)$ and $\Phi(X)$ to be the minimum of the maximum absolute deviation and the minimum of the total square deviations respectively for all parts and products over all partial schedules of X.

The DP recursion for $\o(X)$ is

$$\o(\emptyset) = \o(X{:}X \equiv 0) = 0,$$

$$\o(X) = \min_i\{\max\{\o(X - e_i), \|\Gamma X\|_1\} : i = 1, \dots, n_1, x_i \geq 1\}$$

with $\o(X)$ and $\|\Gamma(X{:}X = d)\|_1 = 0$ for any state X.

The DP recursion for $\Phi(X)$ is

$$\Phi(\emptyset) = \min_i\{\Phi(X - e_i) + (\|\Omega X\|_2)^2, i = 1, \ldots, n_1, x_i \geq 1\}$$

with

$$\Phi(X) \geq 0 \text{ and } (\|\Omega(X: X = d)\|_2)^2 = 0$$

for any state X [41].

In any state X, x_i can have any values $0, 1, \ldots, d_i$. The space and time complexities of the procedures are $O(\prod_{i=1}^{n_1}(d_i + 1))$ and $O(n_1 n \prod_{i=1}^{n_1}(d_i + 1))$, respectively [50]. The number of feasible schedules for any problem instance is $\frac{D_1!}{d_1!, \ldots, d_{n_1}!}$ which is considerably larger than the number of states in the DP recursion. $\prod_{i=1}^{n_1}(d_i + 1) \leq (\frac{D_1+n_1}{n_1})^{n_1}$ shows that the DP algorithm is effective for small number of products even with large number of copies. During the enumeration process, an excessive amount of time or that of space is reduced by using some fast heuristic as a filter which eliminates any state from DP's state space that would lead to no optimality [41]. If the heuristics yield near-optimal sequences, then the state space size could be reduced. The DP algorithm progresses through the state space in the forward direction of increasing the cardinality as the procedure generates all states X with $|X| = k$ before $|X| = k + 1$ for all $k = 1, \ldots, D_1$.

3.3. Assignment method

The problem can be solved pseudo-polynomially transforming the problem into an equivalent assignment problem. Calculation of the assignment costs is based on the level curves $f_i(j - kr_i)$, $j = 0, 1, \ldots, d_i$; $k = 0, 1, \ldots, D$ and the positions in which each copy j of product i, $j = 1, \ldots, d_i$, $i = 1, \ldots, n$ is sequenced.

If all copies of product i are sequenced at their ideal positions $Z_{ij} = \left\lceil\frac{2j-1}{2r_i}\right\rceil$, the ceiling of the unique crossing point satisfying $f_i(j - kr_i) = f_i(j - 1 - kr_i)$, $j = 1, \ldots, d_i$, the product i will contribute the cost $inf_j f_i(j - k r_i)$ to the total cost of the solution and an optimal sequence is obvious. Sequencing the products at their ideal positions minimizes the problems F and G, however, leads to infeasibility when more than one copy compete for the same ideal position in the sequence. Competition occurs in general case. Higher priority is given to j over j whenever $j < j$ to avoid competition and (i, j) is assigned to a position k, $k \neq \left\lceil\frac{2j-1}{2r_i}\right\rceil$.

The new assignment contributes additional cost $C_{ijk} \geq 0$ where

$$C_{ijk} = \sum_{l=k}^{Z_{ij}-1} \psi_{ijl}, \qquad if\ k < Z_{ij},$$

$$\begin{cases} 0, & if\ k = Z_{ij}, \\ \sum_{l=Z_{ij}}^{k-1} \psi_{ijl}, & if\ k > Z_{ij} \end{cases}$$

with

$$\psi_{ijl} = \begin{cases} f_i(j - lr_i) - f_i(j - 1 - lr_i), if\ l < Z_{ij}, \\ f_i(j - 1 - lr_i) - f_i(j - lr_i), if\ l \geq Z_{ij}. \end{cases}$$

The assignment problem equivalent to the problem G is, [40],

$$\min \quad \sum_{k=1}^{D} \sum_{i=1}^{n} \sum_{j=1}^{d_i} C_{ijk} x_{ijk} \tag{3.1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \sum_{j=1}^{d_i} x_{ijk} = 1, \qquad k = 1, \dots, D \tag{3.2}$$

$$\sum_{k=1}^{D} x_{ijk} = 1, \qquad i = 1, \dots, n;\ j = 1, \dots, d_i \tag{3.3}$$

where $x_{ijk} = 1$, if (i, j) is assigned to time unit k and 0, otherwise

Let $\mathbf{X} = \{(i, j, k) | i = 1, \dots, n; j = 1, \dots, d_i; k = 1, \dots, D\}$, be the set of the assignment of (i, j) to k. A set $X \subseteq \mathbf{X}$ is X -feasible if the following constraints hold.
c_1: For each k, $k = 1, \dots, D$, there is exactly one $(i, j), i = 1, \dots, n; j = 1, \dots, d_i$ such that $(i, j, k) \in X$, i.e., exactly one copy is produced at one time unit.
c_2: For each $(i, j), i = 1, \dots, n; j = 1, \dots, d_i$, there is exactly one k, $k = 1, \dots, D$ such that $(i, j, k) \in X$, i.e., each copy is produced exactly once.
c_3: If $(i, j, k), (i, \acute{j}, \acute{k}) \in X$, and $k < \acute{k}$ then $j < \acute{j}$, i.e., lower indices copies are produced earlier.

Constraints c_1 and c_2 are related to the assignment problem. Constraint c_3 imposes an order on copies of a product.

Theorem 3.1 *[40] For any feasible $X \subseteq \mathbf{X}$,*
$G = \sum_{(i,j,k) \in X} C_{ijk} + \sum_{i=1}^{n} \sum_{k=1}^{D} inf_j f_i(j - kr_i)$.

The result becomes an inequality without c_3. An optimal solution cannot be obtained by simply solving the assignment problem since c_3 is not the assignment type.

Theorem 3.2 *[40] If X satisfies c_1 and c_2, then $\dot{X}$ satisfying c_1, c_2, c_3 with $c(X) \geq c(\dot{X})$ can be determined in $O(D)$ time. Moreover, each copy in the sequence $\dot{s}$ from $\dot{X}$ preserves the order that it has in the sequence s from X.*

Since there are D^2 values ψ_{ijl}, D^2 values C_{ijk} and each takes $O(D)$ time to calculate, the Hungarian method takes $O(D^3)$ time to solve the assignment problem with $2D$ nodes. The assignment can be made order preserving in $O(D)$ time. Hence, an optimal solution to the problem G can be obtained in $O(D^3)$ time, see [40]. A number of algorithms solve the assignment problem of the problem G [52].

The approach for the problem G is applicable in every l_p- norm and particular to l_1-norm, see [21].

The corresponding assignment problem equivalent to the problem F is

$$\min \sum_{k=1}^{D} \sum_{i=1}^{n} \sum_{j=1}^{d_i} B_{ijk} x_{ijk} \tag{3.4}$$

subject to the constraints (3.2) and (3.3) where

$$B_{ijk} = \max\{f_i(j-1-(k-1)r_i) - f_i(j-kr_i)\}, i = 1, \dots, n; j = 1, \dots, d_i; k = 1, \dots, D.$$

The assignment costs grow to the left and to the right from the ideal positions $\left\lceil \frac{2j-1}{2r_i} \right\rceil$ in the assignment matrix $[B_{ijk}]$, [10]. One ideal position exists in each row of the matrix, however, there exist two ideal positions in the case of a competition.

The problem is solved by means either of specific bottleneck assignment algorithms or as a sequence of assignment problem with some modifications such as use of a binary matrix instead of the bottleneck assignment matrix and application of bisection search to find the optimal bottleneck value [10]. Optimal solution can be obtained in $O(D^3)$ time.

The bottleneck assignment costs B_{ijk} for which $f_i(x_{ik} - kr_i) < 1$, $i = 1, \dots, n; k = 1, \dots, D$, can be calculated in time $O(nD)$ but it remains open whether the problem can be solved in $O(nD)$. If it exists, it would be better than the existing solution procedures [38].

A cyclic sequence substantially reduces the time complexity. Such sequences exist in the problem G_s [47, 49]. The cyclic sequences are optimal, too. A concatenation s^m of m copies of an optimal sequence s for the instance $(d_1, \dots, d_n)$ of the problem $G_a$is optimal for $(md_1, \dots, md_n)$, $m \geq 1$ [36]. It builds a sequence for a longer time horizon. Such a sequence can be found under the assumption $f_i = f, i = 1, \dots, n$ where f is convex and symmetric with minimum 0 at 0 [11].

3.4. Perfect matching method

The problem F_a is solved by reducing it to an order-preserving perfect matching problem *via* single machine scheduling release/due date decision problem [55]. The perfect matching problem is constructed in a V_1-convex bipartite graph $G = (V_1 \cup V_2, \mathcal{E})$ with $V_2 = \{(i,j) | i = 1, \ldots, n; j = 1, \ldots, d_i\}$, set of the j^{th} copy of product i, $V_1 = \{1, \ldots, D\}$, the starting times and the edge set $\mathcal{E}$ with the earliest starting time $E(i,j)$ and the latest starting time $L(i,j)$for (i,j) defined as $\mathcal{E} = \{(k,(i,j)) | k \in [E(i,j), L(i,j)] \subseteq V_1\}$. For a given bound B and the level curves $|j - kr_i|$, $i = 1, \ldots, n; j = 0, 1, \ldots, d_i; k = 0, 1, \ldots, D$, the values $E(i,j)$ and $L(i,j)$, $i = 1, \ldots, n; j = 1, \ldots, d_i$ are calculated in time $O(D)$ as the unique integers $E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil$ and $L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$ [13].

A modified version of earliest due date (EDD) rule with complexity $O(|E|)$ in V_1-convex bipartite graph $(V_1 \cup V_2, \mathcal{E})$ finds an order-preserving perfect matching for the upper bound $B \leq 1$ [26].

A stronger upper bound has been obtained for the problem F_a. If $\dot{B}$ be any optimal value, then $\frac{1}{\Delta_i}\left\lfloor \frac{\Delta_i}{2} \right\rfloor \leq \dot{B} \leq 1 - \frac{1}{D}$ where $\Delta_i = \frac{D}{\gcd(d_i, D)}$, $i = 1, \ldots, n$ [13], and $\dot{B} \leq 1 - \frac{1}{2(n-1)}$ [62]. Therefore, it holds $\frac{1}{\Delta_i}\left\lfloor \frac{\Delta_i}{2} \right\rfloor \leq \dot{B} \leq 1 - \max\{\frac{1}{D}, \frac{1}{2(n-1)}\}$ for $n \geq 2$. The optimal value $\dot{B}$ cannot be less than $\frac{1}{2}$ for even Δ_i since $\frac{1}{\Delta_i}\left\lfloor \frac{\Delta_i}{2} \right\rfloor = \frac{1}{2}$ and cannot be less than $\frac{1}{3}$ for odd Δ_i since $\frac{1}{3} \leq \frac{1}{\Delta_i}\left\lfloor \frac{\Delta_i}{2} \right\rfloor < \frac{1}{2}$ [20]. It is natural to seek instances with optimal value less than $\frac{1}{2}$.

It has been shown that only the standard instance i.e. the instance with $d_1 \leq \cdots \leq d_n$, $\gcd(d_1, \ldots, d_n) = 1$, has optimal $B = \frac{2^{n-2}-1}{2^n-1} < \frac{1}{2}$ if and only if $d_i = 2^{i-1}, i = 1, \ldots, n$ [35, 14]. It came into existence as the small deviations conjecture [13]. If $< \frac{1}{2}$, all products

must be sequenced in the ideal position $\left\lceil \frac{2j-1}{2r_i} \right\rceil$ for each j, which happens if d_n is divisible by each $d_i, i = 1, \ldots, n-1$. This geometric proof exploits a natural symmetry of regular polygons inscribed in a circle of circumference D such that each polygon corresponds to a different product having d_i corners for product i at $\left\lceil \frac{2j-1}{2r_i} \right\rceil$ points on the perimeter of the circle. Consequently, n demands are the first n non-negative powers of 2 [35].

The small deviations conjecture is shown to be true as a consequence of the Fraenkel's conjecture for symmetric case using a fact that a solution to the problem F_a with $d_i = 2^{i-1}$ for $i = 1, \ldots, n$, $n > 2$ is periodic, symmetric and balanced word [14]. The Fraenkel's conjecture for symmetric case states that a periodic, symmetric and balanced word with $r_1 < \cdots < r_n, n > 2$, exists if and only if $r_i = \frac{2^{i-1}}{2^n-1}$ [14].

A δ-balanced word on a finite set $\{1, \ldots, n\}$ is an infinite sequence $s = s_1 s_2 \ldots$ with $s_i \in \{1, \ldots, n\}$ such that every two subsequences of equal length consist of only those letters whose number of occurrences in each subsequence differ by at most a positive integer δ (See [63]). Note that 1-balanced word is a balanced word. Consider a finite word W on $\{1, \ldots, n\}$ of length D with d_i occurrences of a letter i and $r_i = \frac{d_i}{D}$, the rate of letter i with $r_1 \le \cdots \le r_n$. W is said to be symmetric if $W = W^R$, a mirror reflection of W. An infinite word w is periodic if $w = WW\ldots$ for some W.

For a sequence s with maximum deviation B, any infinite periodic word w, with period s is 1-balanced, 2-balanced and 3-balanced on each product i, if $B < \frac{1}{2}, B < \frac{3}{4}$ and $B < 1$, respectively [29]. The inclusions are proper [21].

Unfortunately, the 1-balanced words are unlikely for most rates to exist. There exists an optimal sequence for the problem F_a in the set of all 3-balanced words. However, it remains unresolved whether there always exists a 2-balanced word that is optimal for the problem F_a. The challenging problem of balanced words in practice is to construct an infinite periodic sequence over a finite set of letters with given rates and distributed as evenly as possible.
Though, only the instance $d_i = 2^{i-1}$, $i = 1, \ldots, n. n > 2$ has $B < \frac{1}{2}$ for the problem F_a, for $n = 2$, infinitely many instances with $B < \frac{1}{2}$ exist i.e. the optimal value of the problem F_a is less than $\frac{1}{2}$ if and only if one of demands d_1 or d_2 is odd and the other even [14]. A sequence with distances $\left\lceil \frac{D}{d_1} \right\rceil$ and $\left\lfloor \frac{D}{d_1} \right\rfloor$ for product 1 with demand d_1 and $\left\lceil \frac{D}{d_2} \right\rceil$ and $\left\lfloor \frac{D}{d_2} \right\rfloor$ for product 2 with demand d_2 is optimal for two product case.

This procedure solves both the problem F_a and the response time variability problem for $n = 2$, which is not true in general for $n > 2$. The response time variability problem minimizes the variability of time for which clients, events, jobs or products wait for the next turn in obtaining the resources necessary for their advance. This problem intends to utilize the resources so as to ensure a fair sharing of common resources between the products which requires to be evenly distributed such that the occurrences in any two consecutive items of the same product is to keep at constant distance as much as possible all the time. The general case of the problem is NP-hard [17]. This result naturally motivates to look at other possible common solutions with respect to different objectives.

The EDD algorithm matches each ascending $k \in V_1$ to the unmatched (i,j) with the smallest $L(i,j)$. Since $E(i,j)$ and $L(i,j)$ are strictly monotonic increasing for consecutive copies of each product [55] and $E(i,j+1)$ cannot be less than $L(i,j)$ with $B < 1$ [37] the algorithm ensures the perfect matching to be order-preserved.

The weighted problem can analogously be reduced to the order-preserved perfect matching problem [57]. Heavy weightage for particular copies of a product restricts the time window $[E(i,j), L(i,j)]$ and increases the separation of consecutive copies of that product in the sequence. $E(i,j)$ and $L(i,j)$ are calculated as the integers $E(i,j) = \left\lceil \frac{jw_i - B}{r_i w_i} \right\rceil$ and $L(i,j) = \left\lfloor \frac{(j-1)w_i + B}{r_i w_i} + 1 \right\rfloor$.

An order-preserved perfect matching gives rise to a feasible solution.

The necessary and sufficient condition for a feasible solution to the problem F_a is the following.

Theorem 3.3 *[13] The problem F_a has a feasible solution if and only if for all $k_1, k_2 \in \{1,\dots,D\}$ with $k_1 \le k_2$ and,*
$\sum_{i=1}^{n} \max(0, \lfloor k_2 r_i + B \rfloor - \lceil (k_1 - 1) r_i - B \rceil) \ge k_2 - k_1 + 1$ *and*
$\sum_{i=1}^{n} \max(0, \lceil k_2 r_i - B \rceil - \lfloor (k_1 - 1) r_i + B \rfloor) \le k_2 - k_1 + 1$.

The theorem tests the feasibility of B in time $O(nD^2)$ though less efficient than $O(D)$ time and of a pair (k_1, k_2), $k_1, k_2 \in \{1,\dots,D\}$ in $O(n)$ time [38].

The perfect matching using a certain bound obtained through a bisection search in the interval $[1 - r_{max}, 1 - \frac{1}{D}]$ yields an optimal sequence in $O(D \log D)$ time. The lower bound $1 - r_{max}$ is tight [55].

Since the deviations are multiples of $\frac{1}{D}$ and the upper bound is $1-\frac{1}{D}$, the bound for the optimal value can be only $B=\frac{k}{D}$ with $k \in \{1,\ldots,D-1\}$ [42]. This fact can be implemented to calculate possible optimal values for the problem F_a only for these values. The optimal sequences of an instance $d_1 = 2$, $d_2 = 3$, $d_3 = 5$ obtained at bound $B=\frac{1}{2}$ are $3-2-1-3-2-3-3-1-2-3$ and $3-2-1-3-3-2-3-1-2-3$, here the 5^{th} and the 6th positions are swapped.

An optimal sequence for the weighted problem is obtained as follows:

Theorem 3.4 *[57] An optimal sequence for the weighted problem can be determined when a bisection search is performed in the interval* $[min_i w_i(1-r_i),\ max_i w_i]$ *in exact pseudopolynomial time* $O(D log(D\phi max_i(w_i)))$, *where* ϕ *is a positive integer constant that depends on the problem data.*

The exact complexity of the problem F_a still remains open. The problem F_a has been proved to be Co-NP but remains open whether it is Co-NP-complete or polynomially solvable [13]. Observation of the input size $O(\sum_{i=1}^{n} log\, d_i) = O(n\, log D)$ and the involvement of nD variables and $O(nD)$ constraints in the model indicate that an expectation of a polynomial algorithm for this problem seems far from trivial.

There exists cyclic optimal sequence for the problem F_a [57]. Let u_i be a factor of D and d_i with $d_i = u_i v_i$ for product i. Each copy of product i is labeled as $(e-1)v_i + j$ where $e = 1,\ldots,u_i$ and $j = 1,\ldots,v_i$, the e^{th} period of copies of product i that consists of v_i copies of product i. There will be u_i such periods for each product. If all of one period's early (late) starting times are calculated, then the early and the late starting times for all copies in all periods can be calculated from these values. When $u_i = gcd(d_i, D) = 1$, the time required to calculate the starting times can be reduced by a factor of 2.

Theorem 3.5 *[57] If* $u = gcd(d_i,\ldots,d_n)$, $i = 1,\ldots,n$, *then the problem* F_a *consists of* u *repetitions of the optimal sequence.*

The problem G can be represented as a complete convex bipartite weighted graph on $V_1 = \{1,\ldots,D\}$. Since each (i,j) can be produced at any instant k, it is clear that $E(i,j) = 1$ and $L(i,j) = D$. The cost C_{ijk} for (i,j) at k is taken as the weight for the edge $(k,(i,j))$. The problem is to find a perfect matching with minimum sum of the weights [56].

Theorem 3.6 *[42] A sequence s for the problem G is optimal if and only if there is a minimum weight perfect matching M with a weight function $w : V_1 \cup V_2 \to R$ such that $w_k + w_{i,j} \le C_{ijk} \forall (k, (i,j)) \in \mathcal{E}$ and $\sum_{k \in V_1} w_k + \sum_{(i,j) \in V_2} w_{ij} = \sum_{(k,(i,j)) \in M} C_{ijk} = s$.*

Let us say an incomplete convex bipartite graph on V_1 if weights are attributed to only those edges $(k, (i,j))$ of which $k \in [E(i,j), L(i,j)]$ with $B \le 1$. This substantially reduces the number of weights to be calculated. A 1-bounded optimal solution for the problem G, if exists, could be obtained in $O(nD^2 \log D)$ time, since $|E| \le (n+2)D$ holds for $B \le 1$ [56].

Theorem 3.7 *[33] The sequence optimal to the problem G with $f_i = f$, $\forall i$, and $-1 \le x_{ik} - kr_i \le 1$ for the incomplete graph is also optimal to the problem G for the complete graph.*

This result cannot be generalized for non-identical cost functions in $[-1, 1]$. As an example, the instance $(24, 24, 28, 28, 42, 42, 42, 42, 48, 16)$ with the cost functions,

$f_1(x) = f_2(x) = \alpha_1 |x|,\ f_3(x) = f_4(x) = \alpha_2 |x|, f_5(x) = f_6(x) = f_7(x) = f_8(x) = \alpha_3 |x|,\ f_9(x) = \alpha_4 |x|,\ f_{10}(x) = \alpha_5 |x|$

where

$\alpha_1 = 168^2.8.7.\alpha_2,\ \alpha_2 = 168^2.6.6.\alpha_3, \alpha_3 = 168^2.4.2.\alpha_4,\ \alpha_4 = 168^2.7.\alpha_5, \alpha_5 = 1$

shows that $-1 \le x_{ik} - kr_i \le 1$ will not hold for some positions [33].

But the existence of such a solution is rarely possible. The question of determining minimum B such that the optimal solution to the problem G is B-bounded remains unanswered [21]. It is shown that the upper bound on the optimal value of the problem G is nD though the bound is not tight. However, the lower bound for the problem G_s is $\sum_{i=1}^{n} \frac{D^2 - d_i^2}{12D}$ [1]. Note that a solution is said to be B-bounded or B-feasible if the deviation is less than a given bound B.

The perfect matching method can also be applied to the generalized pinwheel scheduling problem or the Liu-Layland periodic scheduling in hard real-time environments, see [37]. The generalized pinwheel scheduling problem for n pairs of positive integers $(a_1, b_1), \ldots, (a_n, b_n)$ is to find an infinite sequence $s = s_1 s_2 \ldots$ on finite set $\{1, \ldots, n\}$ such that $s_j \in \{1, \ldots, n\}$, $j \in N$ and any subsequence of s consisting of b_i

consecutive elements of s contains i at least a_i times, $i \in \{1, \dots, n\}$. The solution procedure to the problem F_a with $B < 1$ and the rates $r_i = \frac{a_i+1}{b_i}$, $i = 1, \dots, n$ yields a generalized pinwheel schedule for the instance $(a_1, b_1), \dots, (a_n, b_n)$ if $\sum_{i=1}^{n} \frac{a_i}{b_i} + \leq 1$ [38]. The Liu-Layland periodic scheduling problem is to find an infinite sequence $s = s_1 s_2 \dots$ on a finite set $\{1, \dots, n\}$ such that $s_j \in \{1, \dots, n\}$, $j \in N$ and a preemptive and periodic job i occurs exactly C_i times on any subsequence of s, consisting of T_i consecutive elements of s with $C_i \leq T_i$, $i \in \{1, \dots, n\}$ where C_i and T_i are the run-time and request period for job i. The solution to the problem F_a with $B < 1$ and rates $r_i = \frac{C_i}{T_i}$, $i = 1, \dots, n$ is a periodic schedule [38].

3.5. Simultaneous optimality

Study of finding solutions that minimize a number of objective functions simultaneously is useful. Such solutions not only reduce time complexity of the problem but also are more applicable in practice.

A Pareto algorithm that determines all Pareto optimal sequences for the bicriterion sequencing problem with the objectives F_a and G exist. The algorithm determines an order preserving perfect matching with $B \leq 1$. Then a minimum weight order-preserving perfect matching with the weight C_{ijk} for the edge $(k, (i, j))$, $i = 1, \dots, n;\ j = 1, \dots, d_i;\ k = 1, \dots, D$ is determined. The corresponding production sequence is a Pareto optimal sequence. A Pareto optimal solution can be determined in $O(nD^2 \log D)$ time and all Pareto optimal solutions in $O(nd_{max}D^2 \log D)$ time [56].

Let S_1 be the set of all 1-feasible sequences. The two problems are S_1-equivalent if both have the same set of optimal sequences on S_1. The problems G_a and G_s on S_1 have the same cost $C_{ijk} \geq 0$ for $k \in [E(i, j), L(i, j)]$ [42] and are S_1-equivalent [18, 19]. The assumption in [33] that the S_1-equivalence is due to symmetry and convexity of the objectives is not true. The instance $(23,\ 23,\ 1,\ 1,\ 1,\ 1)$ with the function

$$f(x_{ik} - kr_i) = -\frac{1}{1-\dot{x}}(x_{ik} - kr_i) - \frac{\dot{x}}{1-\dot{x}}$$
$$(x_{ik} - kr_i) \leq -\dot{x} \qquad 0$$
$$-\dot{x} \leq (x_{ik} - kr_i) \leq \dot{x}$$
$$\frac{1}{1-\dot{x}}(x_{ik} - kr_i) - \frac{\dot{x}}{1-\dot{x}}$$
$$\dot{x} \leq (x_{ik} - kr_i)$$

is a counterexample [18], where $\dot{x}$ is optimal value to the problem F_a. An optimal sequence for the problem G_a in S_1 is optimal for the problem G_s in S_1, too. With this, the problem G_s can be solved by means of solving the problem G_a in S_1. It is advantageous for the complexity since the conversion of the floating point numbers to integers of absolute penalties required is smaller in magnitude than that of the square penalties [33]. An optimal solution in S_1 to the problem G_s may not be optimal to the problem G_a, [18, 19]. If the problem G_a has no optimal solution in S_1, the optimality is not guaranteed, however, it provides a lower and an upper bounds for the optimal solution to the problem G_s. The problems G_a and G_s may not have optimal sequences in S_1 [18].

4. The PRV and Apportionment Problems

The apportionment problem, though it appears in different situations, has been studied as a problem for the assignment of seats of a legislature to states or parties and applied in real sense [58, 5, 37]. There exists a connection between the PRVP and the apportionment problem [8, 30].

In divisor method of apportionment, a divisor function Δ, a monotone real-valued function defined over the set of non-negative integers, is defined as $t \leq \Delta(t) \leq t+1$ where t is an integer for which there exists no pair of integers $\dot{t} \geq 0$ and $\ddot{t} \geq 1$ with $\Delta(\dot{t} = \dot{t} + 1)$ and $\Delta(\ddot{t} = \ddot{t})$. Suppose that a cumulative seats x_{ik} have been apportioned in the stages 1 through k. Then a seat is apportioned to a state j in the stage $k+1$ when $\frac{p_j}{\Delta(x_{jk})} \geq \frac{p_i}{\Delta(x_{ik})}, i = 1, \dots, n$ with $p_j > p_i$ implies $\frac{p_j}{0} > \frac{p_i}{0}$, where p_i and p_j are the populations of states i and j respectively. The time complexity of the procedure is $O(nD)$ [4]. A divisor method is said to be parametric if $\Delta(t) = t + \delta \ \forall t, 0 \leq \delta \leq 1$. Adams, Condorcet, Webster and Sainte-Lague and Jefferson and d'Hondt use this method with $\delta = 0$, $\delta = 0.4, \delta =$ 0.5, and $\delta = 1$, respectively [3].

The EDD rule in [28] coincides with the parametric method of apportionment with $\delta = \frac{1}{2}$ [8]. The parametric method is cyclic [3]. The method, developed by [55] to break a tie by choosing the smallest $L(i, j)$ for unmatched (i, j) to sequence at $k \in V_1$ while solving the bottleneck PRVP, is the parametric method of apportionment with $\delta = B < 1$ [30]. The sequences of three products with demands $d_1 = 2,\ d_2 = 3$ and $d_3 = 5$ obtained by parametric method with $\delta = \frac{1}{2}$ are the same the perfect matching method with $B = \frac{1}{2}$ yields.

It is noteworthy that the apportionment problem is more directly related to a problem that determines the number of units of products to produce in such a way that the proportions are as close to the ideal proportions as possible when total number of units are given [8].

5. Production Smoothing Problem

The assumption that allows setup and arbitrary processing times forces the problem to be the production smoothing problem, a two-phase problem. The first phase is the batching problem that determines batch size and the number of batches of the products. The second phase is the sequencing problem that sequences the batches.

A takt-time $t = \frac{\tilde{D}}{D}$, the ratio of the time horizon $\tilde{D}$ to the number of time-buckets D is used as a key factor. A batch (a copy or several copies) of a product is produced during a takt-time. Let $\tilde{d}_i = \left\lceil \frac{d_i}{\left\lceil \frac{\frac{\tilde{D}}{D} - s_i}{p_i} \right\rceil} \right\rceil$, $i = 1, \ldots, n$, batches of product i be produced during $\tilde{D}$ such that $D - \sum_{i=1}^{n} \tilde{d}_i \geq 0$, where s_i and p_i are the setup and the processing time of product $i, i = 1, \ldots, n$.

The multi-objective non-linear integer programming model of the problem [38], is

minimize $\quad \sum_{k=1}^{D} \sum_{i=1}^{n} (\frac{d_i}{\tilde{d}_i})^2 (x_{ik} - k\frac{\tilde{d}_i}{D})^2 \qquad (5.1)$

maximize $\quad D \qquad (5.2)$

subject to

$$x_{i(k-1)} \leq x_{ik} \qquad (5.3)$$

$$x_{i0} = 0, x_{iD} = \tilde{d} \qquad (5.4)$$

$$\frac{\tilde{D}}{max_i\{s_i + p_i\}} \geq D \qquad (5.5)$$

$$D \geq 0, integer \qquad (5.6)$$

The constraints (5.3) and (5.4) show that each product is assigned in $\tilde{d}_i$ batches and the constraints (5.5) and (5.6) ensure the feasibility of D.

Recently, a Pareto optimal solution has been developed [38]. The solution procedure determines D in $O(nD)$ time and sequence of the batches is determined in $O(D^3)$ time by transforming the problem into the assignment problem.

The cost $\tilde{C}_{ijk}$ of assigning j^{th} batch of product i to the k^{th} position is

$$C_{ijk} = (\frac{d_i}{\tilde{d}_i})^2 \sum_{l=k}^{\tilde{Z}_{ij}-1} \tilde{\psi}_{ijl} \quad \text{if } k < \tilde{Z}_{ij},$$

$$0 \quad \text{if } k = \tilde{Z}_{ij}$$

$$(\frac{d_i}{\tilde{d}_i})^2 \sum_{l=\tilde{Z}_{ij}}^{k-1} \tilde{\psi}_{ijl} \quad \text{if } k > \tilde{Z}_{ij}$$

with $\tilde{\psi}_{ijl} = (j - l\frac{\tilde{d}_i}{D})^2 - (j - 1 - l\frac{\tilde{d}_i}{D})^2$, if $l < \tilde{Z}_{ij}$,

$$(j - 1 - l\frac{\tilde{d}_i}{D})^2 - (j - l\frac{\tilde{d}_i}{D})^2, \text{ if } l \geq \tilde{Z}_{ij},$$

and $\left\lceil \frac{(2j-1)D}{2\tilde{d}_i} \right\rceil$, the ceiling of the unique crossing point satisfying

$(j - l\frac{\tilde{d}_i}{D})^2 = (j - 1 - l\frac{\tilde{d}_i}{D})^2, j = 1, \ldots, \tilde{d}_i$.

Some heuristics and meta-heuristics appear in the literature though the batching problem is proven to be NP-hard [66].

A dynamic programming for the exact solution has been explored. However, its use in real environment due to its computational time is impractical [68]. The dynamic programming has been extended to a bounded dynamic programming procedure to solve large-size problem within practical times. Some heuristics, meta-heuristics and hybrid meta-heuristics such as north-east solution search, parametric heuristic search, strategic oscillation, scatter search, path relinking, robust tabu search are also introduced to solve the problem [67]. The problem has been studied on a single machine in [67] and on the flow shop in [68].

6. Smoothing Workload Problem

The problem of smoothing the workload on each workstation on the line is a secondary concern of the MMJITSP. This case has drawn attention from researchers and practitioners as MMJIT sequencing that deals with the goals of keeping a constant rate of parts usage and of smoothing workload.

By t_{is}, we represent the assembly time required for a unit of a product $i, i = 1, \ldots, n$ on a workstation $s, s = 1, \ldots, m$. Clearly, the assembly time required for d_i unites of product i is $d_i t_{is}$. Then, let $T_s = \sum_{i=1}^{n} d_i t_{is}$ be the total assembly time on s over planning horizon T. Let $c = \frac{T}{D}$ be the cycle time, where $D = \sum_{i=1}^{n} d_i$ is the total demand. Ideally, the workstation s should spend $\frac{ck\, d_i t_{is}}{T_s}$ time on product i during the periods 1 through k. However, the actual time required is $t_{is} x_{ik}$, where x_{ik}, $i = 1, \ldots, n,\ k = 1, \ldots, D$ be the cumulative production of product i during the same periods. The deviation between the actual and the ideal assembly times on workstation s incurs either idle line or work overload on the line. The objective of the problem is to minimize the sum of the deviations on all the workstations of the line.

The mathematical model of the smoothing problem [34], is

$$\text{minimize } \sum_{k=1}^{D} \sum_{i=1}^{n} \sum_{s=1}^{m} f_i\left(t_{is} x_{ik} - \frac{ckd_i t_{is}}{T_s}\right)$$

subject to $(2.5)-(2.8)$. f_i is non-negative unimodal convex function having 0 at 0.

The model assumes all products may not have the same operation time at any workstation on the line. The problem to minimize $\sum_{s=1}^{m} f_i\left(t_{is} x_{ik} - \frac{ckd_i t_{is}}{T_s}\right)$ subject to the constraints $(2.5)-(2.8)$ shows that the smoothing workload problem has the same form of the PRVP [34]. A model of the problem similar to the PRVP is also formulated in [46]. The solution to this problem can be obtained using EDD rule in [28]. Another model is in existence based on the concept that the worker of a workstation stops the conveyor of the products if incompletion of the operations occur within the work zone. Two algorithms branch and bound for small size problems and simulated annealing for large size case solve the problem [64].

A pseudo-polynomial solution procedure with complexity $O(logD)$ exist to solve the problem with a finite number of workstations, the displacement time and the time the worker needs to go from one finished product to another one entering the station. See [7]. A tabu search solves the problem with utility work (work done by the utility workers), equivalent to minimizing the work overload, for several products and workstations [53].

A number of papers studies joint problem that simultaneously addresses both parts usage and work load goals. See, for example, [2, 61]. A dynamic programming (DP) is effective for small number of products though with large number of copies [50]. For large problems, two myopic heuristics 'one-stage' heuristic with complexity $O(n)$ that fills one position at a time and its improved case 'two-stage' heuristic with complexity $O(n^2)$ exist [50, 48]. The joint problem as an assignment problem gives rise to optimal sequence for small input size [31].

7. Concluding Remarks

The mathematical models for MMJITSP and different sequencing approaches developed till date have been analyzed. The MMJITSP with the goal of keeping constant rate of usage of parts is focused. The study shows that the problems have real world exciting applications as well as interesting mathematical features of theoretical value. We explicitly explore, with justification of the ground for future research, the questions which still remain open and are challenging.

The problem, under the assumption that the products require approximately the same number and mix of parts or the pegging assumption (single-level) is solvable. A pseudo-polynomial algorithm of the assignment problem is applicable to the problem G. The approach can also be applied to the bottleneck PRVP with necessary modification.

The other approach for solving the problem F_a is the binary search for B-feasible sequence on perfect matching in bipartite graph. It is also of pseudo-polynomial complexity. This property is applicable to other general convex symmetric nonnegative functions also. The bound $B = 1$ is sufficient for searching an optimal solution to the problem F_a and similar results hold for the problem F_s.

The approach applied to the incomplete bipartite graphs to solve the problem G is developed. But it is yet unknown what should be the minimum size of B such that the B-bounded solution guarantees an optimal sequence. Looking for a good B is appealing as this would reduce the complexity.

Despite much effort to solve the PRVP with pseudo-polynomial complexity on the input size of the demands, the exact complexity of the single-level problem still remains open. The problem F_a has been proved to be Co-NP but remains open whether it is Co-NP-complete or polynomially solvable. To have a conclusive statement, it would be one issue of the future research. Analyzing the work-in-progress, solution of this problem with polynomial time complexity seems unlikely to exist.

Since the PRVP is a group of single-objective problems and the properties of optimal sequences may differ significantly for different objective functions, obtaining common or closely related optimal sequences to different objective functions would significantly save the complexity cost.

The ORVP even with two-levels are strongly NP-hard. Therefore, an improvement of existing approximation algorithms, for example dynamic programming or local search techniques would contribute to the research.

Existence of cyclic optimal sequences also considerably reduces the computational time. This problem has been resolved for the PRV case. However, the conjecture whether cyclic sequences to the ORVP are optimal is still open.

The elegant algebraic concept of balanced words introduced in this field is relatively new. The 1-balanced words cannot be obtained for most rates, but the set of all 3-balanced words consists of optimal sequence for the problem F_a. Minimality of this set is unknown and enumeration of this set for optimality is expensive. It is still unsolved whether the set of all 2-balanced words is sufficient for an optimal sequence for the problem F_a. Characterization of balanced words to the other MMJITSP would strengthen the concept of balanced words in obtaining balanced sequence.

The production smoothing problem as a variant of the MMJITSP with arbitrary nonzero processing and setup times helps bridge the gap between the theoretical achievements and industrial practice. Study of this problem in a variety of manufacturing environments for example shop scheduling systems is an important research area.

The relation between the MMJITSP and well established apportionment problem found in the literature shows that the parametric method with $\delta = \frac{1}{2}$ seems to be closely related to Inman and Bulfin's EDD algorithm and the perfect matching method though no formal proof is in existence.

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