



Comparing Numerical Methods for the Solution of the Damped Forced Oscillator Problem

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Abstract

In this paper, we present a comparative study between the Adomian decomposition method and two classical well-known Runge-Kutta and central difference methods for the solution of damped forced oscillator problem. We show that the Adomian decomposition method for this problem gives more accurate approximations relative to other numerical methods and is easier to apply.

Keywords: Adomian decomposition method, differential equation, damped forced oscillator.

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1 Introduction

The topic of the Adomian decomposition method (ADM) has been rapidly growing in recent years. The concept of this method was first introduced by G. Adomian at the beginning of 1980's [1,2]. This method has been applied to a wide class of deterministic and stochastic problems of mathematical and physical sciences [3,4,5,6,7]. The method provides the solution in some rapidly convergent series with components that are elegantly computed. This method can be used to solve all types of linear and nonlinear equations such as differential and integral equations, so it is known as a powerful method. Another important advantage of this method is that it can reduce the size of computations, while increases the accuracy of the approximate solutions.

Several authors have compared the ADM with some existing techniques through solving different types of problems. Wazwaz proposed a new approach to develop a non-perturbed approximate solution for the Thomas-Fermi equation. This approach is based upon a modification of the ADM [8]. In another work, he introduced a comparison between the ADM and the Taylor series method [9]. Advantages of the ADM over the Picard's method have been showed by Rich in [10]. S. M. El-Sayed et al. [12] compared the ADM and Wavelet-Galerkin method through solving the integro-differential equations and showed that the ADM is efficient and easy to use. Edwards et al. [11] have introduced their comparison of ADM and Runge-Kutta method for approximate solutions of some predator prey model equations.

Some reports show that the ADM for linear problems is equivalent to one of the classical methods. N. Bellomo et al. [13] show that using ADM to solve linear integral equations tends to successive approximation method. E. Babolian et al. have shown that solving the system of linear equations in the form $AX = B$ by ADM tends to Jacobi iterative method [15]. While for some problems, the method is equivalent to one of classical methods, we show that for damped forced oscillator, as a well-known linear problem which appears widely in many physical fields and is described by a second order ordinary differential equation, this method gives more accurate approximations of solution than other numerical methods.

In this paper, we outline a reliable comparison between three powerful methods. The first is ADM which our work mainly focused on it. The other applied numerical methods are central difference method (CDM) and Runge-Kutta method (RKM). The results of the ADM are compared with the exact solution and those obtained by RKM and CDM, for different values of constants. It should be noted

that we only report the numerical results of RKM and CDM, and we do not explain these methods.

2 The problem

Damped forced oscillator

Oscillation is the repeated motion of a particle, a body or a system, displaced from its equilibrium position. This motion appears widely in many physical subjects such as acoustics, molecular spectra, vibrations of mechanisms and electrical circuits etc. Oscillating systems are classified according to the forces which act on the system [16,17].

As a special case when a mass, connected to a spring, displaces from its equilibrium position, a restoring force acts on it and makes it to oscillate. For small displacements, according to Hook's law, the force is proportional to displacement, x . If a damping force, proportional to the velocity of the system, dx/dt , acts on the system, the motion is called damped oscillation. Finally exciting the mass by an external time dependent force, $F(t)$, we have a damped forced oscillator. Using Newton's second law, this motion describes by the following equation

$$mx'' = -bx' - kx + F(t),$$

where m is the mass, k is the stiffness of spring, and b is the viscous damping coefficient [17].

Special choices are $F(t) = 2(1 - \sin t)$, $m = 2kg$, $k = 1N/m$, $b = 0.3Ns/m$ and $x(0) = x'(0) = 0$ as initial conditions [18].

3 The methods and solutions

In this section, we first obtain the exact solution for a special case of the problem, introduced in section 2, then approximate the solutions, using three numerical methods. The first applied numerical method is the ADM. The other methods are well known RKM and CDM.

3.1 Exact solution

Using classical methods [19], the exact solution of the equation (1), with introduced coefficients, will be as follows

$$x(t) = e^{-0.075t} (C_1 \cos(0.703118t) + C_2 \sin(0.703118t)) + 2 + \frac{200}{109} \sin t + \frac{60}{109} \cos t$$

applying initial conditions, we have

$$C_1 = -278/109 \text{ and } C_2 = -110425000/38319931.$$

3.2 Adomian decomposition method solution of (1)

Consider the equation (1) with given initial conditions. Denoting d^2/dt^2 by L , we have L^{-1} , as two-fold integration from 0 to t . Therefore this equation can be written as:

$$mLx = -bx' - kx + F(t)$$

Operating with L^{-1} ,

$$L^{-1}Lx = -(b/m)L^{-1}x' - (k/m)L^{-1}x + L^{-1}F(t)$$

$$x = x(0) + tx'(0) - b/mL^{-1}x' - k/mL^{-1}x + 1/mL^{-1}F(t),$$

$$x = x(0) + tx'(0) - b/mL^{-1}d/dt(x) - k/mL^{-1}x + 1/mL^{-1}F(t).$$

To use ADM, let $x = \sum_{m=0}^{\infty} x_n$. Hence, from (3) we have

$$\begin{aligned}
 x_0 &= x(0) + tx'(0) + 1/m L^{-1} F(t) \\
 x_1 &= -b/m L^{-1} d/dt x_0 - k/m L^{-1} x_0, \\
 x_2 &= -b/m L^{-1} d/dt x_1 - k/m L^{-1} x_1, \\
 &\vdots \\
 x_{m+1} &= -b/m L^{-1} d/dt x_m - k/m L^{-1} x_m, \\
 &\vdots
 \end{aligned}$$

In practice, all terms of the series $x(t) = \sum_{m=0}^{\infty} x_m(t)$, cannot be determined, so we use an approximation of the solution by the following truncated series:

$$\varphi_k(t) = \sum_{m=0}^{k-1} x_m(t), \text{ with } \lim_{k \rightarrow \infty} \varphi_k(t) = x(t).$$

For the convergence of the Adomian decomposition scheme, we refer the reader to [1,20,21]. To approximate the solution as a polynomial for the special case introduced in section 2, replacing $F(t)$, with first four terms of its Maclaurin series in equation (5), we obtain

$$\begin{aligned}
 x_0 &= x(t) + tx'(0) + \int_0^t \int_0^t (1-t+t^3/6-t^5/120+t^7/5040) dt dt \\
 &= t^2/2 - t^3/6 + t^5/120 - t^7/5040 + t^9/362880
 \end{aligned}$$

$$\begin{aligned}
 x_1 &= -0.15 \int_0^t \int_0^t d/dt(x_0) dt dt \\
 &= -0.025t^3 - 0.0145833t^4 - 0.000208333t^6 + t^5/240 - t^7/10080 + \dots
 \end{aligned}$$

$$\begin{aligned}
 x_2 &= -0.15 \int_0^t \int_0^t d/dt(x_1) dt dt \\
 &= -0.0009375t^4 - 0.0010625t^5 + 0.000138889t^6 - 0.0000451389t^7 + \dots
 \end{aligned}$$

In a similar manner, the components x_k are calculated for $k = 3, 4, \dots$, but for brevity will not be listed. Considering (6), the approximate solution including six terms is

$$\begin{aligned}
 x \approx \phi_5 &= t^2/2 - 0.191667 t^3 - 0.0136458 t^4 \\
 &\quad + 0.0135344 t^5 - 0.000110929 t^6 - \dots - 3.02201 \times 10^{-25} t^{23}
 \end{aligned}$$

Using this polynomial, the approximations of x values for $t = k\pi/10$ ($k = 0, 1, \dots, 10$), will be obtained as shown at table 1, labeled by ADM1.

More accurate solutions of the problem will be obtained, if we use six terms of Maclaurin series of $F(t)$ in equation (5). The results are shown at table 1 as ADM2.

3.3 Runge-Kutta Method (RKM)

In the Runge-Kutta method, an approximation to dependent variable at $t + \Delta t$ is obtained from it and t in such a way that the power series expansion of the approximation coincides, up to terms of a certain order $(\Delta t)^n$, in the time interval Δt , with the actual Taylor series expansion of $t + \Delta t$ in powers of Δt . The method is based on the assumption that the higher derivatives exist at required points. The main drawback of the method is that each forward step requires several computations of the functions, thus increasing the computational cost. This method extensively described at [18]. The differential equation (1) is solved using 4th order RKM. Table (1) shows the approximate solutions for $0 \leq t \leq \pi$, and $h = \pi/10$, labeled by RKM.

3.4 Central Difference Method (CDM)

In this method the continuous variable t , is replaced by the discrete variable t_i and the differential equation is solved progressively in time increments $h = t_{i+1} - t_i$. The solution obtained is an approximation but suitably selecting the time increment, improves the accuracy of the solutions. In this method, the solution domain replaces with a finite number of points, known as mesh points. The method is based on the Taylor's series expansion of dependent variable, about the grid point i .

For more details, we refer the reader to [18]. Finally using $h = \pi/10$, the systems response will be obtained as shown in table 1, labeled by CDM.

4 Results and Discussions

In this section, we consider the results of three numerical methods to the damped forced oscillator problem. In order to verify the efficiency of the ADM in comparison with exact solutions and the RKM and CDM, we report the numerical solutions for different values of t . Table 1, shows the results of the exact solution and the numerical methods applied to the equation (1) at $[0, \pi]$, with time steps $\pi/10$. The second column shows the results of the exact solution (2). The third and fourth columns show the results of ADM choosing four and six terms of Maclaurin series of $F(t) = 1 - \sin t$, in equation (5). At the fifth and sixth columns you see the results of 4th order RKM and CDM respectively.

Time	Exact	ADM1	ADM2	RKM	CDM
0	0	0	0	0	0
$\pi/10$	0.04331342	0.04331343	0.04331343	0.04329083	0.04934802
$2\pi/10$	0.14902732	0.14902736	0.14902736	0.14898049	0.16067235
$3\pi/10$	0.28266644	0.28266654	0.28266653	0.28259993	0.29887250
$4\pi/10$	0.41548994	0.41549041	0.41549011	0.41541276	0.43471614
$5\pi/10$	0.52766675	0.52767045	0.52766701	0.52758992	0.54806644
$6\pi/10$	0.61013810	0.61016370	0.61013849	0.61007204	0.62977479
$7\pi/10$	0.66500924	0.66514511	0.66500994	0.66496128	0.68207794
$8\pi/10$	0.70447466	0.70505310	0.70447665	0.70444714	0.71750410
$9\pi/10$	0.74844113	0.75051183	0.74844883	0.74843041	0.75645436
π	0.82115244	0.82760961	0.82118323	0.82114892	0.82376858

Table 1. Numerical results of applied methods

To have an accurate comparison between the numerical methods, absolute errors of each numerical method are calculated and shown in table 2.

Time	ADM1-Error	ADM2-Error	RKM-Error	CDM-Error
0	0	0	0	0
$\pi/10$	1.03×10^{-8}	1.03×10^{-8}	2.25×10^{-5}	1.03×10^{-3}
$2\pi/10$	4.25×10^{-8}	4.24×10^{-8}	4.68×10^{-5}	1.03×10^{-2}
$3\pi/10$	1.08×10^{-7}	9.61×10^{-8}	6.65×10^{-5}	1.03×10^{-2}
$4\pi/10$	4.68×10^{-7}	1.69×10^{-7}	7.71×10^{-5}	1.03×10^{-2}
$5\pi/10$	3.70×10^{-6}	2.61×10^{-7}	7.68×10^{-5}	1.03×10^{-2}
$6\pi/10$	2.56×10^{-5}	3.90×10^{-7}	6.60×10^{-5}	1.03×10^{-2}
$7\pi/10$	1.35×10^{-4}	7.06×10^{-7}	4.79×10^{-5}	1.03×10^{-2}
$8\pi/10$	5.78×10^{-4}	1.99×10^{-6}	2.75×10^{-5}	1.03×10^{-2}
$9\pi/10$	2.07×10^{-3}	7.69×10^{-6}	1.07×10^{-5}	1.03×10^{-3}
π	6.45×10^{-3}	3.07×10^{-5}	3.52×10^{-6}	1.03×10^{-3}

Table 2. Absolut errors of the methods

To increase the accuracy of ADM, we used six terms of Maclaurin series of $F(t)$. In other word, using more terms of Maclaurin series, reduces the error of ADM. Table 2, shows that using suitable order of Maclaurin series, the ADM gives more accurate results than RKM while in RKM, changing the steps, gives more accurate results which has computational complexity and is not recommended. Table 2 shows that the CDM has the highest error of the order of 10^{-2} , while the maximum error of the RKM is about 10^{-5} .

5 Conclusion

In this work, we used three numerical methods to approximate the solutions of differential equation, governing on the oscillative systems, and compared the results with the exact solution. The CDM gives the results with an unacceptable error relative to the exact solution. The RKM tends to the results which are suitably accurate relative to the exact solutions, but the ADM gives more accurate results. The study shows that the ADM in a simple way gives the nearest results to the

exact solutions while, the RKM needs to convert the differential equation to a system of differential equations [18], and so has computational complexity. So, we prefer the ADM to RKM, and introduce the ADM as a simple and efficient method.

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