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Well- posedness of the Rothe difference scheme for reverse parabolic equations

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Abstract

We consider the Rothe difference scheme for approximate solution of the abstract parabolic equation in a Hilbert space with the nonlocal boundary condition. Theorems on stability estimates, coercivity and almost coercivity estimates for the solution of this difference scheme are established. In application, new coercivity inequalities for the solution of multi-point nonlocal boundary value difference equations of parabolic type are obtained.

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1 Introduction

In the study of boundary value problems for partial differential equations, the role played by, well-posedness (coercivity inequality) is well known (see Ladyzhenskaya et al. 1968, Ladyzhenskaya and Ural'tseva 1968, Vishik et al. 1959). Coercivity inequalities for nonlocal boundary value problems for partial differential equations parabolic and elliptic types have been studied extensively by many researchers see Aibeche and Favini (2005), Clement and Guerre (1999), Shakhmurov (2004), Sobolevskii (1971) and references given therein.
In the paper (see Ashyralyev and el al. 2008) we considered the abstract nonlocal boundary value problem

$$\begin{cases} \frac{du(t)}{dt} - Au(t) = f(t) \quad (0 \le t \le 1), \\ u(1) = \sum_{k=1}^{p} \alpha_k u(\theta_k) + \varphi, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1 \end{cases} \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator A, under the assumption

$$\sum_{k=1}^{p} |\alpha_k| \le 1 \tag{2}$$

The well-posedness of multi-point nonlocal boundary value problem (1) in spaces $C_1^{\alpha}(H)$ and $C^{\alpha}(H)$ was established. Moreover, as applications, these astract results enabled us to obtain new coercivity estimates in various Hölder norms for the solutions of nonlocal boundary value problems for parabolic equations.

In the present article, our focus is the well-posedness of the first order of accuracy Rothe difference scheme

$$\begin{cases}\tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = \varphi_k, \varphi_k = f(t_k), \\ t_k = k\tau, \ 1 \le k \le N, \ N\tau = 1, \\ u_N = \sum_{m=1}^{p} \alpha_m u_{\ell_m} + \varphi, \\ \ell_m = \left[\dfrac{\theta_m}{\tau}\right], \quad 1 \le m \le p \end{cases} \tag{3}$$

for approximately solving problem (1).

Let $[0,1]_\tau = \{t_k = k\tau, k = 1, \cdots, N, \ N\tau = 1\}$ be the *uniform grid space* with step size $\tau > 0$, where N is a fixed positive integer.

Throughout the paper, $F([0,1]_\tau, H)$ denotes the linear space of grid functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Hilbert space H.

Let $C_\tau(H) = C([0,1]_\tau, H)$ be the Banach space of bounded grid functions with the norm

$$\|\varphi^\tau\|_{C^\alpha(H)} = \|\varphi^\tau\|_{C^\alpha(H)} + \max_{1 \le k < k+r \le N} \frac{\|\varphi_{k+r} - \varphi_k\|_H}{(r\tau)^\alpha},$$

$$\|\varphi^\tau\|_{C^\alpha(H)} = \|\varphi^\tau\|_{C^\alpha(H)} + \max_{1 \le k < k+r \le N} \frac{((N-k)\tau)^\alpha \|\varphi_{k+r} - \varphi_k\|_H}{(r\tau)^\alpha}.$$

We say that difference problem (3) is *stable* in $F([0,1]_\tau, H)$, if we have the following *stability estimate*

$$\left\|\{u_{k-1}\}_1^N\right\|_{F([0,1]_\tau, H)} \le M\left(\|\varphi^\tau\|_{F([0,1]_\tau, H)} + \|\varphi\|_H\right),$$

where M is independent of φ^τ, φ and τ.

Difference problem (3) is said to be *well − posed* in $F([0,1]_\tau, H)$, if for every

$\varphi^{\tau} \in F\left([0,1]_{\tau}, H\right)$ problem (3) is uniquely solvable and have the following *coercivity estimate*:

$$\left\|\left\{\tau^{-1}\left(u_k-u_{k-1}\right)\right\}_1^N\right\|_{F\left([0,1]_\tau,H\right)}+\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{F\left([0,1]_\tau,H\right)}$$
$$\leq M\left(\left\|\phi^{\tau}\right\|_{F\left([0,1]_\tau,H\right)}+\left\|A\phi\right\|_{H'}\right),$$

where $H' \subset H$, M does not depend on φ^{τ}, φ and τ.

Throughout the paper, M shall indicate positive constants which can be different from time to time and we are not interested to precise. We shall write $M(\alpha,\beta,\cdots)$ to stress the fact that the constant depends only on $\alpha,\beta,\cdots$.

2 The First Order of Accuracy Difference Scheme

Let us start with some auxiliary lemmas we need below. Throughout the paper, H denotes a Hilbert space and A is a positive definite self-adjoint operator with $A \geq \delta I$ for some $\delta > 0$.

Lemma 1.1. (See Ashyralyev and Sobolevskii (1994)). The following estimates hold:

$$\left\|t^k A^k e^{-tA}\right\|_{H\to H} \leq M, \quad t>0, \quad K \geq 0, \tag{4}$$

$$\left\|R^k\right\|_{H\to H} \leq \frac{1}{(1+\delta\tau)^k}, \quad K \geq 1, \tag{5}$$

$$\left\|\tau A R^k\right\|_{H\to H} \leq \frac{1}{k}, \quad K \geq 1, \tag{6}$$

$$\left\|A^{\beta}\left(R^{k+r}-R^k\right)\right\|_{H\to H} \leq M\frac{(r\tau)^{\gamma}}{(k\tau)^{\beta+\gamma}}, 1 \leq k < k+r \leq N, \beta \in \{0,1\}, 0 \leq \gamma \leq 1, \tag{7}$$

for some $M, \delta > 0$, which are independent of τ is a positive small number and $R = (I+\tau A)^{-1}$ is the resolvent of A.

Lemma 1.2. Assume that (2) holds. Then, the operator

$$I-\sum_{k=1}^{p}\alpha_k R^{N-\left[\frac{\theta_k}{r}\right]} \tag{8}$$

has an inverse

$$T_\tau = \left(I-\sum_{k=1}^{p}\alpha_k R^{N-\left[\frac{\theta_k}{r}\right]}\right)^{-1}$$

and the following estimate is satisfied:

$$\left\|T_\tau\right\|_{H\to H} \le C\left(\delta,\theta_p\right). \tag{9}$$

Proof. The proof of estimate (9) is based on the triangle inequality, assumption (2), and the estimate

$$\left\|\left(I-\sum_{k=1}^{p}\alpha_k R^{N-\left[\frac{\theta_k}{\tau}\right]}\right)^{-1}\right\|_{H\to H} \le \sup_{\delta\le\mu}\frac{1}{\left|1-\sum_{k=1}^{p}\alpha_k\left(1+\tau\mu\right)^{-N+\left[\frac{\theta_p}{\tau}\right]}\right|}.$$

□

Let us now obtain the formula for the solution of problem (3). It is clear that the first order of accuracy difference scheme

$$\begin{cases}\tau^{-1}\left(u_k-u_{k-1}\right)-Au_{k-1}=\varphi_k,\varphi_k=f\left(t_k\right),\\ t_k=k\tau,\ 1\le k\le N,\ N\tau=1,\\ u_N=\sum_{m=1}^{p}\alpha_m u_{\ell_m}+\varphi,\\ \ell_m=\left[\dfrac{\theta_m}{\tau}\right],\quad 1\le m\le p\end{cases} \tag{10}$$

has a solution and the following formula holds:

$$u_k = R^{N-k}u_N - \sum_{j=k+1}^{N} R^{j-k}\varphi_j\tau, \quad 0 \le k \le N-1. \tag{11}$$

Applying formula (11) and the nonlocal boundary condition

$$\xi = u_N = \sum_{m=1}^{p} \alpha_m u_{\ell_m} + \varphi,$$

we can write

$$\xi = \sum_{k=1}^{p} \alpha_k \left(R^{N-\ell_k}\xi - \sum R^{j-\ell_k}\varphi_j\tau \right) + \varphi.$$

Using Lemma 1.2, we get

$$u_N = T_\tau\left(-\sum_{k=1}^{p} \sum_{j=\ell_k+1}^{N} \alpha_k R^{j-\ell_k}\varphi_j\tau + \varphi \right) \tag{12}$$

Hence, difference equation (10) is uniquely solvable and for the solution, formulas (11) and (12) are valid.

Theorem 1.3. Suppose that (2) holds and $\varphi \in D(A)$. Then, for the solution of difference scheme (10) the following stability estimate

$$\max_{0\le k\le N} \|u_k\|_H \le C(\delta, \theta_p)\left(\|\varphi\|_H + \|\varphi^\tau\|_{C_\tau(H)} \right). \tag{13}$$

holds, where $C(\delta, \theta_p)$ is independent of $\tau, \varphi,$ and φ^τ.

Proof. From estimate (5), formula (11), and $N\tau = 1$ it follows that

$$\max_{0\le k\le N-1} \|u_k\|_H \le \|u_N\|_H + \max_{1\le j\le N} \|\varphi_j\|_H.$$

Using assumption (2), estimate (5), (9), formula (12), and $N_\tau = 1$, we obtain

$$\|u_N\|_H \le C_1(\delta, \theta_p).\left(\|\varphi\|_H + \|\varphi^\tau\|_{C_\tau(H)} \right)$$

From these estimates it follows (13).

This concludes the proof of Theorem 1.3. □

It is well-known that problem (1) in the space $C([0,1],H)$ is not well-posed for the general positive definite self-adjoint operator A and Hilbert space H. Hence, the well-posedness of difference problem (10) in $C([0,1]_\tau, H)$ norm does not take place uniformly with respect to $\tau > 0$.

Theorem 1.4. Let (2) holds and $\varphi \in D(A)$. then, for the solution of difference problem (10), the almost coercivity inequality

$$\begin{aligned} &\left\| \left\{ \tau^{-1}(u_k - u_{k-1}) \right\}_1^N \right\|_{C_\tau(H)} + \left\| \left\{ Au_{k-1} \right\}_1^N \right\|_{C_\tau(H)} \\ &\leq C(\delta, \theta_p) \left(\min\left\{ \ln\frac{1}{\tau}, 1 + \ln\|A\|_{H\to H} \right\} \cdot \left\| \varphi^\tau \right\|_{C\tau(H)} + \|A\varphi\|_H \right) \end{aligned} \tag{14}$$

is valid, where $C(\delta, \theta_p)$ does not depend on τ, φ, and φ^τ.

Proof. Using formula (11), estimate (5), we get for $1 \leq k \leq N$

$$\|Au_{k-1}\|_H \leq \|Au_N\|_H + \|\phi^\tau\|_{C_\tau(H)} \sum_{j=k}^{N} \left\| \tau A R^{j-k+1} \right\|_{H\to H}. \tag{15}$$

It follows from Theorem 1.2 (see Ashyralyev and Sobolevskii (1994) on page 87) that

$$\begin{aligned} \sum_{j=k}^{N} \left\| \tau A R^{j-k+1} \right\|_{H\to H} &= \sum_{m=1}^{N-k+1} \tau \left\| A R^m \right\|_{H\to H} \\ &\leq M \min\left\{ \ln\frac{1}{\tau}, 1 + \left| \ln\|A\|_{H\to H} \right| \right\}. \end{aligned} \tag{16}$$

By formula (12), estimate (9), and assumption (2), we obtain

$$\|Au_N\|_H \leq C(\delta, \theta_p) \left(\|\varphi^\tau\|_{C_\tau(H)} \min\left\{ \ln\frac{1}{\tau}, \left| \ln\|A\|_{H\to H} \right| \right\} + \|A\varphi\|_H \right). \tag{17}$$

Thus, from estimates (15)- (17) it follows that

$$
\begin{aligned}
&\left\| \{Au_{k-1}\}_1^N \right\|_{C_\tau(H)} \le C(\delta,\theta_p) \\
&\times \left(\left\| \varphi^\tau \right\|_{C_\tau(H)} \min\left\{ \ln\frac{1}{\tau}, \left| \ln \|A\|_{H\to H} \right| \right\} + \|A\varphi\|_H \right).
\end{aligned} \tag{18}
$$

Using difference equation (10), the triangle inequality, and estimate (18), we get estimate (14).

This completes the proof of Theorem 1.□

Theorem 1.5. Suppose that (2) holds and $\varphi \in D(A)$**.** Then, the solution of difference scheme (10) satisfy the following stability estimate

$$
\begin{aligned}
&\left\| \{\tau^{-1}(u_k - u_{k-1})\}_1^N \right\|_{C_\tau^\alpha(H)} + \left\| \{Au_{k-1}\}_1^N \right\|_{C_1^\alpha(H)} \\
&\le C(\delta,\theta_p) \left(\frac{1}{\alpha(1-\alpha)} \cdot \left\| \varphi^\tau \right\|_{C_1^\alpha(H)} + \|A\varphi\|_H \right),
\end{aligned} \tag{19}
$$

where $C(\delta,\theta_p)$ is independent of $\tau, \varphi,$ and φ^τ.

Proof. It follows from formula (11) and identity

$$
\tau AR = I - R \tag{20}
$$

that for $1 \le k \le N$

$$
Au_{k-1} = R^{N-k+1}Au_N - \sum_{j=k}^{N} \tau AR^{j-k+1}(\varphi_j - \varphi_{k-1}) + (R^{N-k+1} - I)\varphi_{k-1}. \tag{21}
$$

Thus, using estimate (5), (6), and the definition of $C_1^\alpha(H) - norm$, we get for $1 \le k \le N$

$$
\begin{aligned}
\left\|Au_{k-1}\right\|_H &\le \left\|Au_N\right\|_H \\
&+ \frac{\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}}{\left((N-k+1)\tau\right)^\alpha} \sum_{j=k}^{N} \frac{\tau}{\left((j-k+1)\tau\right)^{1-\alpha}} + 2\left\|\varphi^\tau\right\|_{C_1^\alpha(H)} \qquad (22)\\
&\le \left\|Au_N\right\|_H + \frac{4}{\alpha}\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}.
\end{aligned}
$$

Now, we estimate $\left\|Au_N\right\|_H$.

From formula (12) and $\tau AR = I - R$ it follows that

$$
Au_N = T_\tau\left\{-\sum_{k=1}^{p}\alpha_k\left(\sum_{j=\ell_k+1}^{N}\tau AR^{j-\ell_k}\left(\varphi_j-\varphi_{\ell_k}\right)+\left(I-R^{N-\ell_k}\right)\varphi_{\ell_k}\right)+A_\varphi\right\}.
$$

Hence, by estimates (5), (6), (9), the definition of $C_1^\alpha(H)-norm$, and assumption (2), we obtain

$$
\left\|Au_N\right\|_H \le C\left(\delta,\theta_p\right)\left(\frac{4}{\alpha}\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}+\left\|A\varphi\right\|_H\right). \qquad (23)
$$

Thus, from estimates (22), (23) it follows that

$$
\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{C\tau(H)} \le C\left(\delta,\theta_p\right)\left(\frac{1}{\alpha}\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}+\left\|A\varphi\right\|_H\right). \qquad (24)
$$

Let us now estimate

$$
\max_{1\le k<k+r\le N}\frac{\left((N-k+1)\tau\right)^\alpha\left\|Au_{k-1+r}-Au_{k-1}\right\|_H}{(r\tau)^\alpha}.
$$

First, let $N-k+r\le 2r$. By estimate (24) and the triangle inequality, we obtain

$$
\frac{\left((N-k+1)\tau\right)^\alpha\left\|Au_{k-1+r}-Au_{k-1}\right\|_H}{(r\tau)^\alpha} \le C\left(\delta,\theta_p\right)\left(\frac{1}{\alpha}\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}+\left\|A\varphi\right\|_H\right). \qquad (25)
$$

Next, let $N-k+r\le 2r$. From formula (11) it follows that

$$
\begin{aligned}
Au_{k-1} - Au_{k-1+r} &= \left(R^{N-k+1} - R^{N-k+1-r}\right) Au_N \\
&- \sum_{j=k}^{k+2r-2} \tau AR^{j-k+1}\left(\phi_j - \phi_{k-1}\right) \\
&+ \sum_{j=k+r}^{k+2r-2} \tau AR^{j-(k-1+r)}\left(\phi_j - \phi_{k-1+r}\right) \\
&- \sum_{j=k+2r-1}^{N} \tau A\left(R^{j-k+1} - R^{j-(k-1+r)}\right)\left(\phi_j - \phi_{k-1}\right) \\
&+ \left(I - R^{r-1}\right)\left(\phi_{k-1+r} - \phi_{k-1}\right) + \left(R^{N-k+1} - R^{N-(k-1+r)}\right)\phi_{k-1} \\
&= I_1(k) + I_2(k) + I_3(k) + I_4(k) + I_5(k) + I_6(k).
\end{aligned}
\tag{26}
$$

We first estimate $I_1(k)$. Using estimates (7) for $\beta = 0$, and the fact $N - j + 1 > 2r$, we get

$$
\left\|I_1(k)\right\|_H \le C\left(\delta, \theta_p\right) \frac{(r\tau)^\alpha}{\left((N-k+1)\tau\right)^\alpha}\left(\frac{1}{\alpha}\left\|\varphi^\tau\right\|_{C_1^\alpha(H)} + \left\|A\varphi\right\|_H\right). \tag{27}
$$

Next, it follows from estimate (6) and the definition of $C_1^\alpha(H) - norm$ that

$$
\begin{aligned}
\left\|I_2(k)\right\|_H &\le \frac{\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}}{\left((N-k+1)\tau\right)^\alpha} \sum_{j=k+r}^{k+2r-2} \frac{\tau}{\left((j-(k-1+r))\tau\right)^{1-\alpha}} \\
&\le \frac{2^\alpha}{\alpha} \frac{(r\tau)^\alpha}{\left((N-k+1)\tau\right)^\alpha} \left\|\varphi^\tau\right\|_{C_1^\alpha(H)}.
\end{aligned}
\tag{28}
$$

By using estimate (6), the definition of $C_1^\alpha(H) - norm$, and the fact $N - j + 1 > 2r$, we obtain

$$
\begin{aligned}
\left\|I_3(k)\right\|_H &\le \frac{2^\alpha\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}}{\left((N-k+1)\tau\right)^\alpha} \sum_{j=k+r}^{k+2r-2} \frac{\tau}{\left((j-(k-1+r))\tau\right)^{1-\alpha}} \\
&\le \frac{2^\alpha\left\|\varphi^\tau\right\|_{C_1^\alpha(H)}}{\left((N-k+1)\tau\right)^\alpha} \frac{(r\tau)^\alpha}{\alpha}.
\end{aligned}
\tag{29}
$$

If follows from estimate (7) for $\beta = 1$, the definition of $C_1^{\alpha}(H) - norm$, and the fact $j - k + 1 > 2r$, that

$$\begin{aligned}\left\|I_4(k)\right\|_H &\le M \frac{2^{\alpha}\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}}{((N-k+1)\tau)^{\alpha}} r\tau \sum_{j=k+2r-1}^{N} \frac{\tau}{((j-(k-1+r))\tau)^{2-\alpha}} \\ &\le M \frac{2^{\alpha}\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}}{((N-k+1)\tau)^{\alpha}} \frac{(r\tau)^{\alpha}}{(1-\alpha)}.\end{aligned} \tag{30}$$

Using estimate (5), the definition of $C_1^{\alpha}(H) - norm$, we obtain

$$\left\|I_5(k)\right\|_H \le \frac{2^{\alpha}\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}}{((N-k+1)\tau)^{\alpha}} (r\tau)^{\alpha}. \tag{31}$$

Finally, from estimate (7) for $\beta = 0$ and the fact $N - j + 1 > 2r$ it follows that

$$\left\|I_6(k)\right\|_H \le 2^{\alpha}\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}. \tag{32}$$

Thus, combining estimates (27)-(32), we get for $N - j + 1 > 2r$

$$\frac{((N-k+1)\tau)^{\alpha}\left\|Au_{k-1+r} - Au_{k-1}\right\|_H}{(r\tau)^{\alpha}} \le C(\delta,\theta_p)\left(\frac{\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}}{\alpha(1-\alpha)} + \left\|A\varphi\right\|_H\right). \tag{33}$$

From estimates (25) and (33) it follows that

$$\max_{1\le k<k+r\le N} \frac{((N-k+1)\tau)^{\alpha}\left\|Au_{k-1+r} - Au_{k-1}\right\|_H}{(r\tau)^{\alpha}} \le C(\delta,\theta_p)\left(\frac{\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)}}{\alpha(1-\alpha)} + \left\|A\varphi\right\|_H\right). \tag{34}$$

Combining estimates (24)-(34), we obtain that

$$\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{C\tau(H)} \le C(\delta,\theta_p)\left(\frac{1}{\alpha(1-\alpha)}\left\|\varphi^{\tau}\right\|_{C_1^{\alpha}(H)} + \left\|A\varphi\right\|_H\right). \tag{35}$$

Hence, estimate (19) follows from difference equation (10), estimate (35) and the triangle inequality.

This concludes the proof of Theorem 1.5. □

Let $H_\alpha = H_{\alpha,\infty}(H,A)$ be the fractional space, consisting all $v \in H$ for which the following norm is finite,

$$\|v\|_{H_\alpha} = \|v\|_H + \sup\left\|\lambda^{1-\alpha} Ae^{-\lambda A} v\right\|_H$$

Theorem 1.6. Assume that $\varphi_N - \sum_{k=1}^{p} \alpha_k \varphi_{\ell_k} + A\varphi \in H_\alpha$ and (2). Then, problem (10) is well-posed in $C^\alpha(H)$ and the following coercivity estimate holds

$$\left\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N\right\|_{C^\alpha(H)}$$
$$+\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{C^\alpha(H)} + \left\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N\right\|_{C_\tau(H_\alpha)}$$
$$\le M\left(\frac{1}{\alpha}\left\|\phi_N - \sum_{k=1}^{p}\alpha_k\phi_k + A\phi\right\|_{H_\alpha} + \frac{C(\delta,\theta_p)}{\alpha(1-\alpha)}\left\|\phi^\tau\right\|_{C^\alpha(H)}\right),$$

where M does not depend on φ, φ^τ, and τ.

Proof. Let us establish the estimate for $\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{C^\alpha(H)}$. Similar arguments introduced in the proof of estimate (24) result that

$$\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{C\tau(H)} \le C(\delta,\theta_p)\left(\frac{1}{\alpha}\left\|\varphi^\tau\right\|_{C_1^\alpha(H)} + \left\|A\varphi\right\|_H\right). \qquad (36)$$

Next, we estimate

$$\max_{1\le k<k+r\le N} \frac{\left\|Au_{k-1+r} - Au_{k-1}\right\|_H}{(r\tau)^\alpha}.$$

Using formula (11), we obtain for $1 \le k \le N$ then

$$\begin{aligned} Au_{k-1} &= -\varphi_{k-1} + R^{N-k+1}\left(Au_N + \varphi_N\right) \\ &+ \sum_{j=k}^{N} \tau A R^{j-k+1}\left(\varphi_j - \varphi_{k-1}\right) + R^{N-k+1}\left(\varphi_{k-1} - \varphi_N\right) \\ &= J_1(k) + J_2(k) + J_3(k) + J_4(k). \end{aligned} \tag{37}$$

It is clear that

$$\left\|J_1\right\|_{C^\alpha(H)} = \left\|\varphi^\tau\right\|_{C^\alpha(H)}. \tag{38}$$

Let us estimate $\left\|J_2\right\|_{C^\alpha(H)}$. To alleviate the notation, let $v = \left(Au_N + \varphi_N\right)$. From the definition of $H_\alpha - norm$, the equality

$$\sum_{j=k}^{k+r-1} -\tau A R^{N-j} = R^{N-(k-1+r)} - R^{N-k+1}$$

and the formula connecting the resolvent of the generator of a semigroup with the semigroup it follows that

$$\begin{aligned} &\left\|J_2(k+r) - J_2(k)\right\|_H \\ &\leq \|v\|_{H_\alpha} \int_0^\infty \sum_{j=k}^{k+r-1} \frac{1}{(N-j-1)!} t^{N-j-1} e^{-t} \frac{e^{\frac{-\tau t \delta}{2}} \tau\, dt}{\left(\frac{\tau t}{2}\right)^{1-\alpha}} \\ &\leq 2^{1-\alpha} \|v\|_{H_\alpha} (r\tau)^\alpha \frac{2}{\alpha}. \end{aligned} \tag{39}$$

Thus, using estimate (39), we get

$$\left\|J_2\right\|_{C^\alpha(H)} \leq \frac{4}{\alpha} \|v\|_{H_\alpha}. \tag{40}$$

It follows form estimate (6), the definition of $C^\alpha(H) - norm$, and $N\tau = 1$ that

$$\begin{aligned}\|J_3(k)\|_H &\leq \|\varphi^\tau\|_{C^\alpha(H)} \sum_{j=k}^{N} \frac{\tau}{((j-k+1)\tau)^{1-\alpha}} \\ &\leq \frac{((N-k+1)\tau)^\alpha}{\alpha} \|\varphi^\tau\|_{C^\alpha(H)},\end{aligned} \tag{41}$$

for all k .

Hence, using estimate (41), we obtain

$$\|J_3(k)\|_H \leq \frac{1}{\alpha} \|\varphi^\tau\|_{C^\alpha(H)}. \tag{42}$$

Next, we estimate

$$\max_{1 \leq k < k+r \leq N} \frac{\|J_3(k+r) - J_3(k)\|_H}{(r\tau)^\alpha}.$$

First, let us consider the case $N - j + 1 \leq 2r$. Using the triangle inequality, estimate (41), we get

$$\frac{\|J_3(k+r) - J_3(k)\|_H}{(r\tau)^\alpha} \leq \frac{(2^\alpha + 1)}{\alpha} \|\varphi^\tau\|_{C^\alpha(H)}. \tag{43}$$

Next, we consider the case $N - j + 1 > 2r$. We can write as

$$J_3(k) - J_3(k+r) = J_{31}(k) + J_{32}(k) + J_{33}(k) + J_{34}(k),$$

where $J_{31}(k) = I_2(t), \quad J_{32}(k) = I_3(k), \quad J_{33}(k) = I_4(k)$ (see equation (26)), and

$$J_{34}(k) = \left(R^{r-1} - R^{N-(k-1+r)}\right) \left(\phi_{k-1+r} - \phi_{k-1}\right).$$

So we have

$$\|J_{31}(k)\|_H \leq \frac{2^\alpha (r\tau)^\alpha}{\alpha} \|\varphi^\tau\|_{C^\alpha(H)}, \tag{44}$$

$$\left\|J_{32}(k)\right\|_H \leq \frac{2^{\alpha}(r\tau)^{\alpha}}{\alpha}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}, \tag{45}$$

$$\left\|J_{33}(k)\right\|_H \leq \frac{2^{-1+\alpha}(r\tau)^{\alpha}}{(1-\alpha)}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{46}$$

Finally, using estimate (5) and the definition of $C^{\alpha}(H)-norm$, we get

$$\left\|J_{34}(k)\right\|_H \leq 2(r\tau)^{\alpha}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{47}$$

Hence, it follows from estimates (44)- (47) that for $N-j+1>2r$,

$$\frac{\left\|J_3(k+\tau)-J_3(k)\right\|_H}{(r\tau)^{\alpha}} \leq \frac{M}{\alpha(1-\alpha)}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{48}$$

Combining estimates (43), (48), we get

$$\max_{1\leq k<k+r\leq N}\frac{\left\|J_3(k+r)-J_3(k)\right\|_H}{(r\tau)^{\alpha}} \leq \frac{M}{\alpha(1-\alpha)}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{49}$$

Thus, estimate (42), (49) result that

$$\left\|J_3\right\|_{C^{\alpha}(H)} = \frac{M}{\alpha(1-\alpha)}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{50}$$

Using estimate (5) and the definition of $C^{\alpha}(H)-norm$, we obtain

$$\left\|J_4(k)\right\|_H \leq \left((N-k+1)\tau\right)^{\alpha}\left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)} \leq \left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}, \quad for \quad all \quad k. \tag{51}$$

Hence, estimate (51) gives

$$\left\|J_4(k)\right\|_H \leq \left\|\varphi^{\tau}\right\|_{C^{\alpha}(H)}. \tag{52}$$

By using estimates (5) , (7) for $\beta=0$, we get for all $1\leq k<k+r\leq N$

$$
\begin{aligned}
\left\|J_4(k+r)-J_4(k)\right\| &\le \left\|R^{N-(k-1+r)}-R^{N-k+1}\right\|_{H\to H}\left\|\varphi_{k-1+r}-\varphi_N\right\|_H \\
&+\left\|R^{N-(k-1+r)}\right\|_{H\to H}\left\|\varphi_{k-1+r}-\varphi_{k-1}\right\|_H \\
&\le (M+1)(r\tau)^\alpha\left\|\varphi^\tau\right\|_{C^\alpha(H)}.
\end{aligned} \tag{53}
$$

So, from estimate (53) it follows that

$$
\max_{1\le k<k+r\le N}\frac{\left\|J_4(k+r)-J_4(k)\right\|_H}{(r\tau)^\alpha}\le M_1\left\|\varphi^\tau\right\|_{C^\alpha(H)}. \tag{54}
$$

Thus, by combining estimates (52), (54), we obtain

$$
\left\|J_4\right\|_{C^\alpha(H)}=M_1\left\|\varphi^\tau\right\|_{C^\alpha(H)}. \tag{55}
$$

From estimates (36), (38), (40), (50), and (55) it results that

$$
\left\|\left\{Au_{k-1}\right\}_1^N\right\|_{C^\alpha(H)}\le M\left(\frac{1}{\alpha}\left\|Au_N+\varphi_N\right\|_{H_\alpha}+\frac{1}{\alpha(1-\alpha)}\left\|\varphi^\tau\right\|_{C^\alpha(H)}\right). \tag{56}
$$

Hence, using the triangle inequality, estimate (56), and difference equation (10), we get

$$
\begin{aligned}
&\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\right\|_{C^\alpha(H)} \\
&\le M\left(\frac{1}{\alpha}\left\|Au_N+\varphi_N\right\|_{H_\alpha}+\frac{1}{\alpha(1-\alpha)}\left\|\varphi^\tau\right\|_{C^\alpha(H)}\right).
\end{aligned} \tag{57}
$$

Let us now establish the estimate for $\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\right\|_{C^\alpha(H)}$.

It results from formula (11) and difference equation (10) that for all k,

$$
\begin{aligned}
\frac{u_k-u_{k-1}}{\tau} &= R^{N-(k-1+r)}(Au_N+\varphi_N)+R^{N-(k-1+r)}(\varphi_k-\varphi_N) \\
&-\sum_{j=k}^{N}\tau AR^{j-(k-1)}(\varphi_j-\varphi_{k-1}) \\
&= G_1(k)+G_2(k)+G_3(k).
\end{aligned}
$$

Using estimate (5) and the definition of $H_\alpha - norm$, we obtain

$$\|G_1(k)\|_{H_\alpha} \leq \|Au_N + \varphi_N\|_{H_\alpha}. \tag{58}$$

Now, using the definition of $H_\alpha - norm$ and the formula connecting the resolvent of the generator of a semigroup with the semigroup, we get

$$\begin{aligned}\|G_2(k)\|_{H_\alpha} &= \sup_{\lambda>0}\left\|\lambda^{1-\alpha}Ae^{-\lambda A}R^{N-k+1}(\varphi_k - \varphi_N)\right\|_H \\ &= \sup_{\lambda>0}\left\|\lambda^{1-\alpha}Ae^{-\lambda A}\int_0^\infty \frac{t^{N-k}e^{-t}}{(N-k)!}e^{-\tau tA}(\varphi_k - \varphi_N)dt\right\|_H \\ &\leq \|\varphi^\tau\|_{C^\alpha(H)}.\end{aligned} \tag{59}$$

Next, let us estimate $\|G_3(k)\|_{H_\alpha}$. Let $\lambda > 0$. From estimates (4), (5), (7) for $\beta = 1$, and identity (20) it follows that

$$\left\|\tau Ae^{-\lambda A}AR^{j-k+1}\right\|_{H\to H} \leq \min\left\{\frac{\tau}{((j-k)\tau)^2}, \frac{\tau}{\lambda^2}\right\} \leq M\frac{\tau}{((j-k)\tau+\lambda)^2}. \tag{60}$$

Using estimate (60) and the definition of $C^\alpha(H) - norm$, we get

$$\begin{aligned}\|G_3(k)\|_{H_\alpha} &\leq M\|\varphi^\tau\|_{C^\alpha(H)}\sup_{\lambda>0}\lambda^{1-\alpha}\sum_{j=k}^{N}\frac{\tau}{((j-k)\tau+\lambda)^{2-\alpha}} \\ &\leq M\frac{\|\varphi^\tau\|_{C^\alpha(H)}}{1-\alpha}.\end{aligned} \tag{61}$$

Hence, combining estimates (58)- (61), we obtain

$$\begin{aligned}&\left\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N\right\|_{C^\alpha(H)} \\ &+ \left\|\{Au_k\}_1^N\right\|_{C^\alpha(H)} + \left\|\left\{\tau^{-1}(u_k - u_{k-1})\right\}_1^N\right\|_{C^\alpha(H)} \\ &\leq M\left(\frac{1}{\alpha}\|Au_N + \varphi_N\|_{H_\alpha} + \frac{1}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C^\alpha(H)}\right).\end{aligned} \tag{63}$$

Now, we estimate $\left\|Au_N + \varphi_N\right\|_{H_\alpha}$. Using formula (12), we get

$$
\begin{aligned}
Au_N + \phi_N &= T_\tau \{-\sum_{k=1}^{p} \alpha_k \sum_{j=\ell_{k+1}}^{N} \tau A R^{j-\ell_k} \left(\phi_j - \phi_{\ell_k}\right) \\
&\quad + \sum_{k=1}^{p} \alpha_k R^{N-\ell_k} \left(\phi_{\ell_k} - \phi_N\right) \\
&\quad + \phi_{N} - \sum_{k=1}^{p} \alpha_k \phi_{\ell_k} + A\phi \} \\
&= P_1 + P_2 + P_3.
\end{aligned}
$$

It follows from estimates (6), (9), (60), assumption (2), and the definition of $C^\alpha(H)-norm$ that

$$\left\|P_1\right\|_{H_\alpha} \le \frac{C(\delta,\theta_p)}{(1-\alpha)} \left\|\varphi^\tau\right\|_{C^\alpha(H)}, \tag{64}$$

$$\left\|P_2\right\|_{H_\alpha} \le C(\delta,\theta_p) \left\|\varphi^\tau\right\|_{C^\alpha(H)}, \tag{65}$$

$$\left\|P_3\right\|_{H_\alpha} \le C(\delta,\theta_p) \left\|\varphi_N - \sum_{k=1}^{p} \alpha_k \varphi_{\ell_k} + A\varphi\right\|_{H_\alpha}. \tag{66}$$

Therefore, estimates (63), (64)- (66) finishes the proof of Theorem 1.6. □

3 Application

In this section, we consider applications of Theorem 1.5 and Theorem 1.6.

First, let us consider the nonlocal boundary value problem for one dimentional parabolic equation

$$\begin{cases} u_t + (a(x)u_x)_x - \delta_u = f(t,x), \quad 0 < t < 1, \quad 0 < x < 1, \\ u(1,x) = \sum_{m=1}^{p} \alpha_m u(\theta_m, x) + \varphi(x), \quad 0 \le x \le 1, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1, \\ u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad 0 \le t \le 1 \end{cases} \tag{67}$$

under assumption (2), where $\delta > 0$, $a(x) \ge a > 0 \quad (x \in (0,1)), \varphi(x)(x \in [0,1])$ and $f(t,x) \quad (t, x \in [0,1])$ are smooth functions.

The discretization of problem (67) is carried out in two steps. In the first step, we define the grid space

$$[0,1]_h = \{x = x_n \quad : \quad x_n = nh, \quad 0 \le n \le M, \quad Mh = 1\}.$$

Let us introduce the Hilbert space $L_{2h} = L([0,1]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ defined on $[0,1]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x \in [0,1]_h} |\varphi(x)|^2 h \right)^{\frac{1}{2}}.$$

To the differential operator A generated by problem (67), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,n} + \delta\varphi_n\}_1^{M-1} \tag{68}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} \dfrac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad 0 < t < 1, \quad x \in [0,1]_h, \\ u^h(1,x) = \sum\limits_{m=1}^{p} \alpha_m u^h(\theta_m, x) + \varphi(x), \quad x \in [0,1]_h, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{69}$$

In the second step, we replace (69) with the difference scheme (10)

$$\begin{cases} \dfrac{u_k^h(x) - u_{k-1}^h(x)}{\tau} - A_h^x u_{k-1}^h(x) = f_k^h(x), \\ f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, \quad 1 \le k \le N, \quad x \in [0,1]_h, \\ u_N^h(x) = \sum\limits_{m=1}^{p} \alpha_m u_{\ell_m}^h(x) + \varphi(x), \quad x \in [0,1]_h, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{70}$$

Theorem 2.1. Let τ and h be sufficiently small numbers. Then, the solutions of difference scheme (70) satisfy the following coercivity stability estimate:

$$\left\| \left\{ \tau^{-1}\left(u_k^h - u_{k-1}^h\right) \right\}_1^N \right\|_{C_1^\alpha\left([0,1]_\tau, L_{2h}\right)} + \left\| \left\{ u_{k-1}^h \right\}_1^N \right\|_{C_1^\alpha\left([0,1]_\tau, W_{2h}^2\right)}$$
$$\le C(\delta, \theta_p) \left(\frac{1}{\alpha(1-\alpha)} \left\| \left\{ f_k^h \right\}_1^N \right\|_{C_1^\alpha\left([0,1]_\tau, W_{2h}^2\right)} + \left\| \varphi^h \right\|_{W_{2h}^2} \right)$$

hold where $C(\delta, \theta_p)$ is independent of τ, $f_k^h(x)$, and $\varphi^h(x)$, $1 \le k \le N-1$.

Theorem 2.2. Let

$$A_h^x \varphi^h(x) = f_1^h(x) - \sum_{k=1}^{p} \alpha_m f_{\ell_m}^h(x).$$

Then, for solutions of the problem (70), we have the following stability inequalities

$$\left\|\left\{\tau^{-1}\left(u_k^h - u_{k-1}^h\right)\right\}_1^N\right\|_{C^\alpha\left([0,1]_\tau, L_{2h}\right)} + \left\|\left\{u_{k-1}^h\right\}_1^N\right\|_{C^\alpha\left([0,1]_\tau, W_{2h}^2\right)}$$
$$\leq \frac{C\left(\delta,\theta_p\right)}{\alpha\left(1-\alpha\right)}\left\|\left\{f_k^h\right\}_1^N\right\|_{C^\alpha(H)},$$

where M does not depend on φ,φ^τ, and τ.

The proof of Theorem 2.1, Theorem 2.2 is based on the abstract Theorem 1.5, Theorem 1.6 and the symmetry properties of the difference operator A_h^x defined by formula (68).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $R^n = \{x = (x_1,\cdots,x_n):\ \ 0 < x_i < 1,\ \ i = 1,\cdots,n\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1]\times\Omega$, the boundary value problem for the multi-dimensional parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \displaystyle\sum_{r=1}^{n}\left(a_r(x)u_{x_r}\right)_{x_r} = f(t,x), \\ x = (x_1,\cdots,x_n)\in\Omega, \quad 0 < t < 1, \\ u(1,x) = \displaystyle\sum_{i=1}^{p}\alpha_i u(\theta_i,x) + \varphi(x), \quad x\in\Omega, \\ 0 \leq \theta_1 < \theta_2 < \cdots < \theta_p < 1, \\ u(t,x) = 0, \quad x\in S, \quad 0 \leq t \leq 1 \end{cases} \tag{71}$$

under assumption (2) is considered. Here $a_r(x)$, $(x\in\Omega),\varphi(x)$ $(x\in\overline{\Omega})$, and $f(t,x)$ $(t\in(0,1),\ \ x\in\Omega)$ are given smooth functions and $a_r(x)\geq a > 0$.

The discretization of problem (71) is carried out in two steps.

In the first step, define the grid space $\widetilde{\Omega}_h = \{x = x_m = (h_1m_1,\cdots,h_nm_n);\ m =$

$(m_1,\cdots,m_n),\ \ 0 \leq m_r \leq N_r,\ \ h_rN_r = 1, r = 1,\cdots,n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega,$

$$S_h = \widetilde{\Omega}_h \cap S.$$

Let L_{2h} denoted the Hilbert space

$$L_{2h} = L_2\left(\widetilde{\Omega}_h\right) = \left\{ \varphi^h(x) : \left(\sum_{x \in \widetilde{\Omega}_h} \left| \varphi^h(x) \right|^2 h_1 \cdots h_2 \right)^{\frac{1}{2}} < \infty \right\}.$$

The differential operator A in (71) is replaced with

$$A_h^x u^h(x) = -\sum_{r=1}^{n} \left(\alpha_r(x) u_{\bar{x}_r}^h \right)_{x_r, j_r}, \tag{72}$$

where the difference operator A_h^x is defined on those grid functions $u^h(x) = 0$, for all $x \in S_h$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}.

Using (71), we get

$$\begin{cases} \dfrac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad 0 < t < 1, \quad x \in \widetilde{\Omega}_h, \\ u^h(1,x) = \sum_{m=1}^{p} \alpha_m u^h(\theta_m, x) + \varphi(x), \quad x \in \widetilde{\Omega}_h, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases} \tag{73}$$

From (73) it follows that

$$\begin{cases} \dfrac{u_k^h(x) - u_{k-1}^h(x)}{\tau} - A_h^x u_{k-1}^h(x) = \varphi_k^h(x), \\ \Omega_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, \quad 1 \le k \le N, \quad x \in \widetilde{\Omega}_h, \\ u_N^h(x) = \sum_{m=1}^{p} \alpha_m u_{\ell_m}^h(x) + \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\ 0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 1. \end{cases}$$

Theorem 2.3. Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solutions of difference scheme (74) satisfy the following coercivity

$$\left\| \left\{ \tau^{-1} \left(u_k^h - u_{k-1}^h \right) \right\}_1^N \right\|_{C_1^\alpha \left([0,1]_\tau , L_{2h} \right)} + \left\| \left\{ u_{k-1}^h \right\}_1^N \right\|_{C_1^\alpha \left([0,1]_\tau , W_{2h}^2 \right)}$$

$$\leq C\left(\delta, \theta_p \right) \left(\frac{1}{\alpha(1-\alpha)} \left\| \left\{ f_k^h \right\}_1^N \right\|_{C_1^\alpha \left([0,1]_\tau , W_{2h}^2 \right)} + \left\| \phi^h \right\|_{W_{2h}^2} \right),$$

where $C(\delta, \theta_p)$ is independent of τ, $f_k^h(x)$, and $\varphi^h(x)$, $1 \leq k \leq N-1$.

Theorem 2.4. Let $A_h^x \varphi^h(x) = \varphi_1^h(x) - \sum_{k=1}^{p} \alpha_m \varphi_{\ell_m}^h(x)$. Then, for solutions of problem (74), we have the following stability inequalities

$$\left\| \left\{ \tau^{-1} \left(u_k^h - u_{k-1}^h \right) \right\}_1^N \right\|_{C^\alpha \left([0,1]_\tau , L_{2h} \right)} + \left\| \left\{ u_{k-1}^h \right\}_1^N \right\|_{C^\alpha \left([0,1]_\tau , W_{2h}^2 \right)}$$

$$\leq \frac{C\left(\delta, \theta_p \right)}{\alpha(1-\alpha)} \left\| \left\{ f_k^h \right\}_1^N \right\|_{C^\alpha (H)}$$

where M does not depend on φ^h, f_k^h, h, and τ.

The proof of Theorem 2.3, Theorem 2.4 is based on the abstract Theorem 1.5, Theorem 1.6, and the symmetry properties of the difference operator A_h^x, defined by formula (72), and the following theorem:

Theorem 2.5. (see Sobolevskii (1975)) For the solutions of the elliptic differential problem

$$\begin{cases} A_h^x u^h(x) = w^h(x), & x \in \widetilde{\Omega}_h, \\ u^h(x) = 0, & x \in S_h, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| \left(u_k^h \right)_{\bar{x}_r \bar{x}_r , j_r} \right\|_{L_{2h}} \leq M \left\| w^h \right\|_{L_{2h}}.$$

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